Algebraic Topology

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Preface

Contents

1 Introduction

In Math 5400, you learned about fundamental group $\pi_1(X)$ of a topological space X which classifies loops in (connected components of) X up to homotopy equivalence. This is the most important invariant of manifolds in low dimension which even uniquely characterizes surfaces

(2-dimensional manifolds). If X is a CW or simplicial complex (including manifolds), the fundamental group only depends on the 2-skeleton of X. Thus, it is far from being sufficient for studying manifolds of dimension three and higher. In this course we learn about higher dimensional homology and homotopy theories that help us study topological/geometric spaces using tools from (homological) algebra (and category theory). Each of these topological invariants are defined for certain classes of topological spaces and some of them are equivalent when restricted to the class of manifolds.

Exercise 1.1. If X is a simplicial or CW complex (c.f. Definitions 2.3 and **??** below), prove that $\pi_1(X)$ only depend on the 2-skeleton of X.

As we will learn in the following sections, higher homotopy groups $\pi_n(X)$ are defined via continuous maps from the n-dimensional cube

$$I^n := [0,1]^n$$

or the n-dimensional ball/disk

$$B^n = B^n(1) = \{v \in \mathbb{R}^n \colon\ |v| \leq 1\}$$

to X which are trivial on the boundary, up to homotopy equivalence. Therefore, for a reason similarly to Exercise 1.1, $\pi_n(X)$ only depends on the $(n+1)$-skeleton of a CW complex X. While the definition of $\pi_n(X)$ is a relatively straightforward generalization of the definition of $\pi_1(X)$, the higher homotopy groups are (often) hard to compute and have properties that are unintuitive. For instance, $\pi_n(S^2) \neq 0$ for infinitely many $n > 2$.

The first homology group $H_1(X, \mathbb{Z})$ is the abelianization of $\pi_1(X)$. More generally, the homology groups $H_n(X; \mathbb{Z})$, with $n \geq 0$, are defined combinatorially/algebraically by viewing balls as simplices and gluing their boundaries consistently instead of collapsing it to a point as is done in the definition of homotopy groups. In other words, in singular homology vs. homotopy, the suppression of boundary is exchanged with a boundary map, resulting in very different structures. Similarly to homotopy groups, the homology group $H_n(X; \mathbb{Z})$ also only depends on the $(n+1)$-skeleton of a CW/Simplicial complex X; however, $H_n(X; \mathbb{Z})$ vanishes for n larger than the dimension of X and is always a commutative ring (over $\mathbb{Z}$). The drawback is that some information is lost; sometimes, we need homotopy groups to make finer conclusions.

In the following sections, we learn about the definition of simplicial/singular/cellular homology, work on some examples and applications (e.g. Brouwer degree of a map), and learn a main tool for computation (i.e., the Mayer-Vietoris long exact sequence). Then, we repeat this process for cohomology and homotopy groups.

2 Simplicial Homology

Abstractly speaking, an m-simplex Δ is the convex hull of $m+1$ generic points in $\mathbb{R}^n$ for some $n \geq m$. More precisely, an m-simplex is a set of the form

$$\Delta_{[v_0,\ldots,v_m]} := \left\{ x_0 v_0 + \ldots + x_m v_m \colon x_i \geq 0, \quad \sum_{i=1}^{m} x_i = 1 \right\} \subset V, \tag{2.1}$$

where V is a vector space of dimension greater than or equal to m and $v_0, \ldots, v_m$ are $(m+1)$ vectors in V such that $\{v_1 - v_0, \ldots, v_m - v_0\}$ is a linearly independent set of vectors. The simplex is so-named because it represents the simplest possible polytope in any given space. For example, a 0-simplex is a point, a 1-simplex is a line segment, a 2-simplex is a triangle, and a 3-simplex is a tetrahedron. The boundary $\partial\Delta$ of any m-simplex Δ is a union of $m+1$ $(m-1)$-simplices, each one being the convex hull of m of the $m+1$ points[1]. We call them codimension-1 faces of Δ. In the notation of (2.1), we have

$$\partial\Delta_{[v_0,\ldots,v_m]} = \bigcup_{i=0}^{m} \Delta_{[v_0,\ldots,v_{i-1},\widehat{v_i},v_{i+1},\ldots v_m]}, \tag{2.2}$$

where $\widehat{v_i}$ means the i-th vector/point is removed from the list. Inductively, we define a codimension-k face of Δ to be a boundary component of a codimension-$(k-1)$ face. In other words, a codimension k face is the convex hull of $m-k+1$ of the original $m+1$ points. From a differential topology point of view, we may think of an m-simplex Δ as a smooth manifold with boundaries and corners such that the corners correspond to codimension 2 and higher faces. Topologically, an m-simplex Δ is homeomorphic to a ball in $\mathbb{R}^m$.

Given two m-simplices $\Delta_{[v_0,\ldots,v_m]}$ and $\Delta_{[v_0',\ldots,v_m']}$, with $v_0, \ldots, v_m \in V$ and $v_0', \ldots, v_m' \in V'$, any linear transformation $h\colon V \longrightarrow V'$ that maps $\{v_0, \ldots, v_m\}$ to $\{v_0', \ldots, v_m'\}$ gives a linear identification of $\Delta_{[v_0,\ldots,v_m]}$ and $\Delta_{[v_0',\ldots,v_m']}$. A linear identification

$$h\colon \Delta_{[v_0,\ldots,v_m]} \longrightarrow \Delta_{[v_0',\ldots,v_m']}$$

is uniquely identified by the one-to-one correspondence of the vertices. Up to linear identification, there is a unique m-simplex in every dimension $m \geq 0$. For instance, all m-simplices can be identified with the standard m-simplex

$$\Delta_{\mathrm{std}} = \Big\{(x_1, \ldots, x_m)\colon x_i \geq 0, \quad \sum_{i=1}^{m} x_i \leq 1\Big\} = \Delta_{[0,e_1,e_2\ldots,e_m]} \subset \mathbb{R}^m,$$

where $e_1, \ldots, e_m$ is the standard basis of $\mathbb{R}^m$.

Thinking of $\Delta_{[v_0,\ldots,v_m]}$ as a manifold, the tangent space of $\Delta_{[v_0,\ldots,v_m]}$ at every point is the span of $\{v_1 - v_0, \ldots, v_m - v_0\}$; i.e.,

$$T_x\Delta_{[v_0,\ldots,v_m]} \cong \mathbb{R}\cdot(v_1 - v_0) \oplus \cdots \oplus \mathbb{R}\cdot(v_m - v_0) \subset V \qquad \forall\, x \in \Delta_{[v_0,\ldots,v_m]}. \tag{2.3}$$

The decomposition on the righthand side determines an orientation on $\Delta_{[v_0,\ldots,v_m]}$ that depends on the ordering $(v_0, \ldots, v_m)$ of these $(m+1)$-vectors. Unless otherwise mentioned, we let $\Delta_{[v_0,\ldots,v_m]}$ to denote the oriented m-simplex with the orientation specified above. For instance, the orientation on $\Delta_{\mathrm{std}} = \Delta_{[0,e_1,e_2\ldots,e_m]}$ is the standard orientation on $\mathbb{R}^m$.

Exercise 2.1. If $\sigma\colon \{0, 1, \ldots, m\} \longrightarrow \{0, 1, \ldots, m\}$ is a permutation, the m-simplices $\Delta_{[v_0,\ldots,v_m]}$ and $\Delta_{[v_{\sigma(0)},\ldots,v_{\sigma(m)}]}$ are the same as sets. How do the orientations of $\Delta_{[v_0,\ldots,v_m]}$ and $\Delta_{[v_{\sigma(0)},\ldots,v_{\sigma(m)}]}$ compare?

[1]There are $\binom{m+1}{m} = m+1$ ways of choosing m points from a set of size $m+1$

In general, if M is an orientable (smooth) manifold and $N \subset \partial M$ is a boundary component of M, then N is also orientable. Given an orientation on M, there are different conventions for orienting N. We opt the following one. The inclusion $N \subset M$ gives an exact sequence of vector bundles

$$0 \longrightarrow TN \longrightarrow TM|_N \longrightarrow \mathcal{N}_M N \longrightarrow 0 \tag{2.4}$$

where $\mathcal{N}_M N$ is the normal line bundle of N in M. It is easy to show that $\mathcal{N}_M N$ is isomorphic to the trivial bundle $N \times \mathbb{R}$. We can choose an isomorphism such that the vector filed corresponding to constant section 1 of $N \times \mathbb{R}$ is an outward-pointing vector field n_{out} along N. The exact sequence above and the choice of an outward-pointing vector field n_{out} gives an isomorphism (a splitting of the short-exact sequence (2.4))

$$TM|_N \cong \big(N \times \mathbb{R}\big) \oplus TN. \tag{2.5}$$

We choose the orientation on TN such that (2.5) is an oriented isomorphism. We call it the induced orientation on N.

By the discussion of the last paragraph, an orientation on an m-simplex Δ induces an orientation on each boundary $(m-1)$-simplex $\Delta' \subset \partial\Delta$.

Exercise 2.2. By (2.2) and with the choice of the orientation in (2.3), how does the boundary orientation on $\Delta_{[v_0,\ldots,v_{i-1},\hat{v_i},v_{i+1},\ldots v_m]} \subset \partial\Delta_{[v_0,\ldots,v_m]}$ compare to the intrinsic orientation of $\Delta_{[v_0,\ldots,v_{i-1},\hat{v_i},v_{i+1},\ldots v_m]}$?

Every codimension 2 face Δ'' is the intersection of two boundary components Δ'_1 and Δ'_2. The boundary orientations induced by Δ'_1 and Δ'_2 on Δ'' are the opposite. This is the key observation in the definition of the singular homology below.

Definition 2.3. A Δ-complex or a simplicial complex $\mathcal{K}$ is a topological space (with the quotient topology) obtained by gluing a collection of simplices along their faces using linear identifications. A pure or homogeneous simplicial m-complex $\mathcal{K}$ is a simplicial complex where the largest dimension of any simplex in $\mathcal{K}$ equals m and every simplex of dimension $n < m$ is a codimension $m - n$ face of some m-simplex $\Delta \in \mathcal{K}$.

Note that the definition of a simplicial complex is purely combinatorial. Also note that by the definition of quotient topology and the fact that the gluing maps are linear isomorphisms, for each simplex Δ in $\mathcal{K}$, the characteristic map $\sigma\colon \Delta \longrightarrow \mathcal{K}$ is continuous (but not necessarily an embedding). For instance, by identifying the two end-points of a 1-simplex (i.e. an interval $[0,1]$) we obtain S^1. The map quotient $[0,1] \longrightarrow S^1$ is not an embedding.

Definition 2.4. If $\mathcal{K}$ is a simplicial complex, the n-skeleton of $\mathcal{K}$ denoted by $\mathcal{K}^{(n)}$ is the union of all simplices of dimension n or less.

By definition, $\mathcal{K}^{(m)} \subset \mathcal{K}^{(n)}$ for all $0 \leq m \leq n$. The 0-skeleton is a discrete space, and the 1-skeleton is a graph. The skeletons of a space are used in obstruction theory, to construct spectral sequences by means of filtrations, and generally to make inductive arguments. They are particularly important when X has infinite dimension, in the sense that the $\mathcal{K} \neq \mathcal{K}^{(n)}$ for all $n \geq 0$.

Remark 2.5. There is a minor difference between the precise definition of a simplicial complex vs. a Δ-complex which has no effect on the computation of simplicial homology. Definition 2.3

is that of a Δ-complex. In the precise definition of a simplicial complex, each n-simplex is uniquely determined by its vertices; in other words, each n-simplex has $n+1$ distinct vertices. As a result, in a simplicial complex, for each simplex Δ in $\mathcal{K}$, the characteristic map $\sigma\colon \Delta \longrightarrow \mathcal{K}$ is an embedding. In a Δ-complex, however, it is allowed to have two triangles that share the same vertices, or a loop consisting of one point and one edge as in the example above.

Exercise 2.6. Show that every Δ-complex admits a subdivision to a finer Δ-complex that is simplicial in the sense of Remark 2.5.

Every manifold M admits a triangulation; i.e., it is homeomorphic to a pure simplicial complex $\mathcal{K}_M$. Figure 1 illustrates a (complicated) triangulation of the 2-torus; one can triangulate a 2-torus with only 2 triangles. In the literature, there are different types of triangulations on a

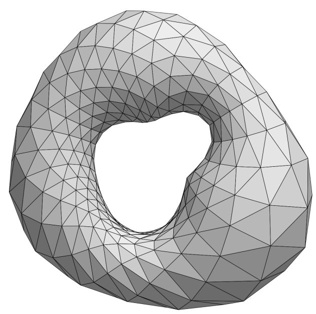

Figure 1: A triangulation of 2-torus (image courtesy of wikipedia)

topological space. Smooth manifolds admit the best kind of triangulation. We refer to [2] for a quick overview of the results.

Theorem 2.7 (Cairns[1]-Whitehead [3])**.** *Every smooth manifold admits an (essentially unique) compatible piecewise linear structure (i.e., a maximal atlas in which the transition maps are piecewise linear), and therefore a (combinatorial) triangulation.*

If M is compact, $\mathcal{K}_M$ has only finitely many faces. If M is oriented, each m-simplex Δ in $\mathcal{K}_M$ inherits an orientation from the orientation on M. In dimensions less than four, Theorem 2.7 holds for topological manifolds (because every topological 3-manifold admits a smooth structure). The problem in higher dimensions is somehow related to a similar issue appearing in Remark 3.5 below.

Exercise 2.8. Find a metrizable topological space that is not homeomorphic to a simplicial complex.

Next, we define and study the simplicial homology groups of a simplicial complex. The definition of simplicial homology generalizes to singular homology which is defined for an arbitrary topological space. The former is much easier for calculations when dealing with simplicial complexes. Let $\mathcal{K}$ be a simplicial complex and $\mathcal{R}$ be a commutative ring. A simplicial k-chain with coefficients in $\mathcal{R}$ is a finite formal sum

$$\tau = \sum c_i \Delta_i$$

where each $c_i \in \mathcal{R}$ and Δ_i is an oriented k-simplex in $\mathcal{K}$. In this definition, we declare that each oriented simplex is equal to the negative of the simplex with the opposite orientation. The group of k-chains on $\mathcal{K}$ is denoted $C_k(\mathcal{K},\mathcal{R})$. This is a free abelian group which has a basis in one-to-one correspondence with the set of k-simplices in $\mathcal{K}$. To define a basis explicitly, one has to choose an orientation of each simplex. If, for instance, M is an oriented m-manifold, for m-simplices in $\mathcal{K}_M$, we usually choose the orientation induced by the orientation on M. For each oriented k-simplex Δ in $\mathcal{K}$, the boundary $\partial\Delta \in C_{k-1}(\mathcal{K},\mathcal{R})$ is the formal sum of the codimension-1 faces with the induced boundary orientation. In other words,

$$\partial\Delta_{[v_0,\dots,v_m]} = \sum_{i=0}^{m} \varepsilon_{i,m}\ \Delta_{[v_0,\dots,v_{i-1},\hat{v_i},v_{i+1},\dots v_m]},$$

where the sign factors $\varepsilon_{i,m} \in \{\pm 1\}$ are calculated in Exercise 2.2. The boundary operator ∂ linearly extends to all simplicial k-chains in $C_k(\mathcal{K},\mathcal{R})$:

$$\partial\colon C_k(\mathcal{K},\mathcal{R}) \longrightarrow C_{k-1}(\mathcal{K},\mathcal{R}).$$

By the last line before Definition 2.3, we have $\partial^2 = 0$; therefore, the so-called simplicial homology groups

$$H_k^{\mathrm{simp}}(\mathcal{K},\mathcal{R}) := \frac{Z_k(\mathcal{K},\mathcal{R}) := \ker\big(\partial\colon C_k(\mathcal{K},\mathcal{R}) \longrightarrow C_{k-1}(\mathcal{K},\mathcal{R})\big)}{B_k(\mathcal{K},\mathcal{R}) := \mathrm{Image}\big(\partial\colon C_{k+1}(\mathcal{K},\mathcal{R}) \longrightarrow C_k(\mathcal{K},\mathcal{R})\big)}, \qquad \forall\ k \geq 0,$$

with coefficients in $\mathcal{R}$ are well-defined.

Exercise 2.9. If $\mathcal{K} = \Delta_{[v_0,\dots,v_m]}$ is the trivial simplicial complex made of only one simplex of dimension m, prove that

$$H_k^{\mathrm{simp}}(\mathcal{K},\mathcal{R}) = \begin{cases} \mathcal{R} & \text{if} \quad k = 0; \\ 0 & \text{otherwise}. \end{cases} \tag{2.6}$$

Example 2.10. We can give $M = S^1$ the structure of a pure 1-complex with one 1-simplex $I = [0,1]$ and one 0-simplex $p = \{1\} \cong \{0\}$. Therefore, taking the coefficients from $\mathcal{R} = \mathbb{Z}$, we have $C_1 = \mathbb{Z}\cdot I$, $C_0 = \mathbb{Z}\cdot p$, and

$$\partial\colon C_1 \longrightarrow C_0, \qquad I \longrightarrow \{1\} - \{0\} = p - p = 0.$$

We conclude that $H_1^{\mathrm{simp}}(S^1,\mathbb{Z}) \cong \mathbb{Z}$ is generated by the homology class of the 1-cycle I, and $H_0^{\mathrm{simp}}(S^1,\mathbb{Z}) \cong \mathbb{Z}$ is generated by the homology class of the 0-cycle p.

Example 2.11. A 2-torus T is the topological space obtained by identifying the vertical and horizontal edges of the square $I^2 = [0,1]\times[0,1]$ (in an orientation-preserving manner) as shown in Figure 2. The square I^2 can be divided into two triangles A and B with the counter clock-wise orientation. Taking the coefficients from $\mathcal{R} = \mathbb{Z}$, the group of 2-chains is generated by A and B, the group of 1-chains is generated by a, b, and c with the chosen orientations as in Figure 2, and the group of 0-chains is generated by the point p. Furthermore, the boundary maps

$$0 \longrightarrow C_2 = \mathbb{Z}\cdot A \oplus \mathbb{Z}\cdot B \xrightarrow{\partial} C_1 = \mathbb{Z}\cdot a \oplus \mathbb{Z}\cdot b \oplus \mathbb{Z}\cdot c \xrightarrow{\partial} C_0 = \mathbb{Z}\cdot p \longrightarrow 0$$

are given by

$$\partial A = c - a + b; \quad \partial B = a - b - c, \qquad \partial a = \partial b = \partial c = p - p = 0.$$

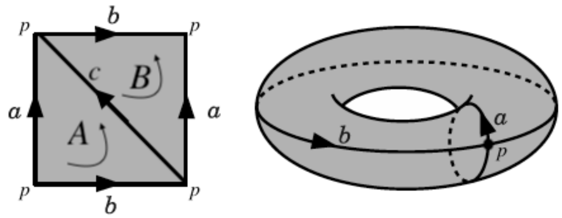

Figure 2: 2-torus obtained from identifying the edges of a rectangle.

Therefore,

$$
\begin{aligned}
H_2^{\text{simp}}(T,\mathbb{Z}) &= \operatorname{Ker}(\partial\colon C_2 \longrightarrow C_1) = \mathbb{Z}\cdot(A+B) \cong \mathbb{Z};\\
H_1^{\text{simp}}(T,\mathbb{Z}) &= \frac{\operatorname{Ker}(\partial\colon C_1 \longrightarrow C_0)}{\operatorname{Im}(\partial\colon C_2 \longrightarrow C_1)} = \frac{C_1}{\mathbb{Z}\cdot(a-b-c)} \cong \mathbb{Z}\cdot a \oplus \mathbb{Z}\cdot b \cong \mathbb{Z}^2;\\
H_0^{\text{simp}}(T,\mathbb{Z}) &= \frac{C_0}{\operatorname{Im}(\partial\colon C_1 \longrightarrow C_0)} = \frac{C_0}{0} = \mathbb{Z}\cdot p \cong \mathbb{Z}.
\end{aligned}
$$

Exercise 2.12. Calculate the homology groups of the Klein bottle and the real projective plane $\mathbb{RP}^2$.

Exercise 2.13. Show that $H_n^{\text{simp}}(\mathcal{K},\mathcal{R})$ only depends on the $(n+1)$-skeleton on $\mathcal{K}$; i.e.,

$$H_n^{\text{simp}}(\mathcal{K},\mathcal{R}) = H_n^{\text{simp}}(\mathcal{K}^{(n+1)},\mathcal{R}) \qquad \forall\ n \geq 0.$$

3 Singular Homology

If a topological space X, such as a manifold, can be given the structure of simplicial complex, it can be done in infinitely many ways. At least, one can start from a simplicial complex and consistently subdivide the simplices into smaller simplices to obtain a more refined simplicial complex. Thus it is a natural question whether simplicial homology of X depends on the choice of the Δ-complex structure on X? In other words, if two simplicial complexes are homeomorphic, do they have isomorphic homology groups? In order to extend the construction of the previous section to an arbitrary topological space X, and answer such questions, we consider arbitrary continuous maps from simplices into X and define singular homology in the following sense.

There are two possible ways to define a singular n-simplex. They lead to slightly different chain complexes but the same homology groups. The first definition below seems to be the one considered in Hatcher. However, in certain parts of the books, it is somewhat unclear whether he is following Definition 3.1 or Definition 3.2.

Definition 3.1. (Definition one) A singular n-simplex in a topological space X is a continuous map $\sigma\colon \Delta \longrightarrow X$ from an oriented n-simplex Δ to X. The boundary of σ, denoted by $\partial\sigma$, is the formal sum of the singular $(n-1)$-simplices obtained from the restrictions of σ to the codimension-1 faces of Δ (with the boundary orientation). We say $\sigma\colon \Delta \longrightarrow X$ and $\sigma'\colon \Delta' \longrightarrow X$ are equal or write $\sigma = \sigma'$ if there is a (orientation-perserving) linear identification $h\colon \Delta \longrightarrow \Delta'$ such that $\sigma \circ h = \sigma'$.

Definition 3.2. (Definition two) A singular n-simplex in a topological space X is a continuous map $\sigma\colon \Delta_{[v_0,\dots,v_n]} \longrightarrow X$ where the orientation considered on $\Delta_{[v_0,\dots,v_n]}$ is the canonical orientation (2.3). The boundary of σ, denoted by $\partial\sigma$, is the formal sum

$$\sum_{i=0}^{m}(-1)^i(\sigma|_{[v_0,\dots\widehat{v_i}\dots v_n]}).$$

We say $\sigma\colon \Delta_{[v_0,\dots,v_n]} \longrightarrow X$ and $\sigma'\colon \Delta_{[w_0,\dots,w_n]} \longrightarrow X$ are equal or write $\sigma = \sigma'$ if $\sigma\circ h = \sigma'$ where $h\colon \Delta_{[w_0,\dots,w_n]} \longrightarrow \Delta_{[v_0,\dots,v_n]}$ is the unique linear identification that maps w_i to v_i, for all $i = 0,\dots,n$.

Given a commutative ring $\mathcal{R}$, for $n \geq 0$, let $C_n(X,\mathcal{R})$ denote the free abelian group generated (over $\mathcal{R}$) by (finite formal sums of) all singular n-simplices. For $n < 0$, define $C_n(X,\mathcal{R})$ to be the trivial group (zero). Compared to $C_n(\mathcal{K},\mathcal{R})$ in the previous section, the groups $C_n(X,\mathcal{R})$ are monstrous because every arbitrary continuous perturbation of a given singular n-simplex $\sigma\colon \Delta \longrightarrow X$ yields a different singular n-simplex. The set of such perturbations is uncountably large.

Remark 3.3. The difference between Definition 3.1 and Definition 3.2 is the following. If $\delta \in \mathrm{S}_{n+1}$ is a permutation of the indices $0, 1, \dots, n$, let

$$h_\delta\colon \Delta_{[v_0,\dots,v_n]} \longrightarrow \Delta_{[v_{\delta(0)},\dots,v_{\delta(n)}]} = \Delta_{[v_0,\dots,v_n]} \tag{3.1}$$

denote the unique linear identification that maps v_i to $v_{\delta(i)}$, for all $i = 0,\dots,n$. Note that $\Delta_{[v_{\delta(0)},\dots,v_{\delta(n)}]}$ and $\Delta_{[v_0,\dots,v_n]}$ are the same simplices with different ordering of the vertices. In Definition 3.1, the singular simplices $\sigma\colon \Delta_{[v_0,\dots,v_n]} \longrightarrow X$ and $(-1)^{\mathrm{sign}(\delta)}\sigma\circ h_\delta\colon \Delta_{[v_0,\dots,v_n]} \longrightarrow X$ are defined to be the same. In Definition 3.2, however, they define different n-simplices unless $\delta = \mathrm{id}$. Therefore, the chain complex arising from Definition 3.2 is larger and admits a projection map to the chain complex arising from Definition 3.1. It can be shown that every element in the kernel of this projection map is a boundary class. More precisely, it can be shown that chain map

$$\sigma \longrightarrow (-1)^{\mathrm{sign}(\delta)}\sigma\circ h_\delta$$

is chain homotopic to the identity map; see Page 211 of Hatcher. Therefore, the singular homology defined using either of these two definitions is the same. Definition 3.1 is somehow more intuitive and easier to use when dealing with simplicial complexes. We will, however, need to use Definition 3.2 in Section 10 to define cup product. It seems impossible to use Definition 3.1 for defining cup product.

An element of $C_n(X,\mathcal{R})$ is called a singular n-chain. The boundary operator ∂ in Definition 3.1 linearly extends to all singular n-chains in $C_n(X,\mathcal{R})$. The extension, called the boundary operator, is an $\mathcal{R}$-linear map

$$\partial\colon C_n(X,\mathcal{R}) \longrightarrow C_{n-1}(M,\mathcal{R}).$$

By the last line before Definition 2.3, we have $\partial^2 = 0$; therefore, the so-called singular homology groups

$$H_n^{\mathrm{sing}}(X,\mathcal{R}) := \frac{Z_n(X,\mathcal{R}) := \ker\big(\partial\colon C_n(X,\mathcal{R}) \longrightarrow C_{n-1}(X,\mathcal{R})\big)}{B_n(X,\mathcal{R}) := \mathrm{Image}\big(\partial\colon C_{n+1}(X,\mathcal{R}) \longrightarrow C_n(X,\mathcal{R})\big)} \qquad \forall\, n \geq 0,$$

with coefficients in $\mathcal{R}$ are well-defined. An element in $Z_n(X,\mathcal{R})$ is called a singular n-cycle and an element in $B_n(X,\mathcal{R})$ is called a singular n-boundary.

Since the abelian groups $C_n(X,\mathcal{R})$ are monstrous, even if X is a nice topological space such as a smooth manifold, it is not clear why $H_n^{\text{sing}}(X,\mathcal{R})$ should be finite. If X is path-connected, it is fairly easy to show that $H_0^{\text{sing}}(X,\mathcal{R})\cong\mathcal{R}$.

Remark 3.4. If we need to clearly distinguish the simplicial and singular chain complexes associated to a topological space X with a Δ-complex structure $\mathcal{K}_X$, we will denote them by $C_n^{\text{sing}}(X,\mathcal{R})$ and $C_n^{\text{simp}}(X_{\mathcal{K}},\mathcal{R})$.

Remark 3.5. Suppose X is a topological space and $f\colon M\longrightarrow X$ is a continuous map from a closed[2] oriented n-dimensional (smooth) manifold M into X. Then f defines a singular homology class $[f]\in H_n^{\text{sing}}(X,\mathbb{Z})$ in the following way. Fix a triangulation $M=\bigcup_{\alpha\in\mathcal{I}}\Delta_\alpha$ of M which gives M the structure of a pure n-dimensional simplicial complex $\mathcal{K}_M$ with oriented n-simplices $\{\Delta_\alpha\}_{\alpha\in\mathcal{I}}$. Note that $\mathcal{I}$ is finite because M is compact. Also, because $\partial M=\emptyset$, every $(n-1)$-simplex in $\mathcal{K}_M$ either connects two different Δ_α or it appears as two boundary components of the same Δ_α attached to each other. The last sentence implies that

$$\partial\sum_{\alpha\in\mathcal{I}}\sigma_\alpha=0,$$

where $\sigma_\alpha=f|_{\Delta_\alpha}\colon\Delta_\alpha\longrightarrow X$ is the singular n-simplex obtained by the restriction of f to Δ_α. Therefore, $\sum_{\alpha\in\mathcal{I}}\sigma_\alpha$ defines a homology class $[f]\in H_n(X,\mathbb{Z})$. In particular, every closed oriented n-dimensional submanifold $M\subset X$ of a manifold X has a well-defined homology class $[M]\subset H_n(X,\mathbb{Z})$. The following lemma shows that the converse is true in degrees $0,1,2$. In degrees 3 and higher, the proof of Lemma 3.6 may produce a topological space M that does not admit a smooth structure (it may have codimension 3 and higher singularities).

Lemma 3.6. *For $n\geq 0$, every element in the singular homology group $H_n^{\text{sing}}(X,\mathbb{Z})$ of a topological space X is represented by a continuous map $f\colon\mathcal{K}\longrightarrow X$ such that $\mathcal{K}$ is a pure compact n-dimensional simplicial complex and $\mathcal{K}-\mathcal{K}^{(n-3)}$ has the structure of an oriented n-manifold without boundary. In particular, elements of $H_1(X,\mathbb{Z})$ are represented by collections of oriented loops in X, and elements of $H_2(X,\mathbb{Z})$ are represented by maps of closed oriented surfaces.*

A representative $[f]$ of any homology class $H_n^{\text{sing}}(X,\mathbb{Z})$ as in Lemma 3.6 is often called a pseudo-cycle; see [4, Theorem 1.1].

Proof. A singular n-chain can be written as

$$\sigma\colon\sum_{\alpha\in\mathcal{I}}\Delta_\alpha\longrightarrow X$$

where $\mathcal{I}$ is a finite index set, Δ_α is an oriented n-simplex, and $\sigma_\alpha:=\sigma|_{\Delta_\alpha}$ is a continuous map into X. Note that one exchange a multiple $m_\alpha\sigma_\alpha$ of σ_α by m_α copies of σ_α in the summation and change the orientation of the Δ_α to account for negative coefficients. If $\partial\sigma=0$, for every $\alpha\in\mathcal{I}$ and every codimension-1 face $\Delta_{\alpha;\beta}\subset\Delta_\alpha$, there is $\beta\in\mathcal{I}$ and a codimension-1 face $\tau_\beta\subset\Delta_\beta$ such that $\sigma_\alpha|_{\tau_\alpha}$ cancels with $\sigma_\beta|_{\tau_\beta}$; i.e. $\sigma_\alpha|_{\tau_\alpha}+\sigma_\beta|_{\tau_\beta}=0$ or equally

$$\sigma_\alpha|_{\tau_\alpha}=\sigma_\beta|_{-\tau_\beta}. \tag{3.2}$$

[2]closed means compact without boundary.

By Definition 3.1, we can glue Δ_α and Δ_β along τ_α and τ_β using an orientation-reversing linear identification $h\colon \tau_\alpha \longrightarrow \tau_\beta$ to obtain an oriented manifold (with boundary and corners) $\Delta_\alpha \cup_h \Delta_\beta$ (that includes $\tau_\alpha \sim_h \tau_\beta$ as a hypersurface). By (3.2), the map

$$\begin{aligned} &\sigma_{\Delta_\alpha \cup_h \Delta_\beta}\colon \Delta_\alpha \cup_h \Delta_\beta \longrightarrow X, \\ &\sigma_{\Delta_\alpha \cup_h \Delta_\beta}|_{\Delta_\alpha} = \sigma_\alpha, \qquad \sigma_{\Delta_\alpha \cup_h \Delta_\beta}|_{\Delta_\beta} = \sigma_\beta, \end{aligned}$$

is well-defined and continuous. Continuing this process for all canceling boundary components of $\partial\sigma$, we obtain a pure n-dimensional simplicial complex $\mathcal{K}$ made of n-simplices $\{\Delta_\alpha\}_{\alpha\in\mathcal{I}}$ and a continuous function $f\colon \mathcal{K} \longrightarrow X$ such that $f|_{\Delta_\alpha} = \sigma_\alpha$. The claim is that $\mathcal{K}$ has the structure of an oriented smooth compact manifold without boundary away from the codimension-3 skeleton; i.e. $\mathcal{K} - \mathcal{K}^{(n-3)}$ admits the structure of a smooth manifold. Orientability of $\mathcal{K} - \mathcal{K}^{(n-3)}$ follows from the fact that the gluing maps h are orientation reversing. That $\mathcal{K} - \mathcal{K}^{(n-3)}$ has no boundary follows from the fact that codimension-1 faces of $\{\Delta_\alpha\}_{\alpha\in\mathcal{I}}$ come in pairs and for each pair $\tau_\alpha \cong \tau_\beta$, $\Delta_\alpha \cup_h \Delta_\beta$ includes $\tau_\alpha \sim_h \tau_\beta$ as a hypersurface. Compactness is a consequence of the finiteness of $\mathcal{I}$.

If p is an interior point of Δ_α, the charts defining the (smooth) manifold structure on $\mathcal{K} - \mathcal{K}^{(n-3)}$ are given by the affine structure on $\Delta_\alpha \subset \mathbb{R}^n$. If p is a point in the interior of the hypersurface $\tau_\alpha \sim_h \tau_\beta$, p is obtained by identifying an interior point $x \in \tau_\alpha$ and an interior point $y \in \tau_\beta$ via the linear identification h. A sufficiently small neighborhood of x in Δ_α has the form $(-\epsilon, 0] \times U$ where U is a neighborhood of 0 in $\mathbb{R}^{n-1}$. Similarly, a sufficiently small neighborhood of y in Δ_β is of the form $[0, \epsilon) \times U$ where U is a neighborhood of 0 in $\mathbb{R}^{n-1}$. Therefore, a sufficiently small neighborhood of p in $\Delta_\alpha \cup_h \Delta_\beta$ is of the form $(-\epsilon, \epsilon) \times U$ where U is a neighborhood of 0 in $\mathbb{R}^{n-1}$. The last identification defines the desired smooth chart around p. Furthermore, since h is orientation reversing, these identifications can be chosen to be compatible with the standard orientations on $(-\epsilon, 0] \times U, [0, \epsilon) \times U \subset \mathbb{R}^n$.

Finally, suppose $p \in \mathcal{K}^{(n-2)} - \mathcal{K}^{(n-3)}$. Then the n-simplices containing p have the following arrangement. There is sequence of indices $\alpha_1, \alpha_2, \ldots, \alpha_k$, with $\alpha_{k+1} \coloneqq \alpha_1$, such that Δ_{α_i} and $\Delta_{\alpha_{i+1}}$ are glued along codimension-1 faces $\tau_{\alpha_i,\mathrm{out}} \subset \Delta_{\alpha_i}$ and $\tau_{\alpha_{i+1},\mathrm{in}} \subset \Delta_{\alpha_{i+1}}$ via a linear identification h_i. For each i, the intersection $\delta_i = \tau_{\alpha_i,\mathrm{in}} \cap \tau_{\alpha_i,\mathrm{out}}$ is a codimension-2 face of Δ_i. The codimension-2 faces δ_i have the same image $\delta \in \mathcal{K}^{(n-2)}$ and p is an interior point of the $(n-2)$-simplex δ. A sufficiently small neighborhood of p in each Δ_{α_i} is the form $C_i \times U$ where U is a neighborhood of 0 in $\mathbb{R}^{n-2}$ and C_i is a neighborhood of 0 in a non-trivial cone in $\mathbb{R}^2$ between the vectors v_i and u_i; i.e.

$$0 \in C_i \subset \{tv_i + su_i \colon s, t \geq 0\} \subset \mathbb{R}^2.$$

The linear identification h_i maps $\mathbb{R} \cdot u_i$ to $\mathbb{R} \cdot v_{i+1}$ in a cyclic order. Therefore, a sufficiently small neighborhood of p in each $\mathcal{K}$ is the form $C \times U$ where U is a neighborhood of 0 in $\mathbb{R}^{n-2}$ and C is a neighborhood of 0 in a 2-dimensional cone. Every such C is homeomorphic to a disk. Furthermore, we can choose the angles between v_i and u_i (smooth structure on C_i) such that identification of C and a disk is smooth (the total angle is 2π). This gives the desired smooth charts around the points of $\mathcal{K}^{(n-2)} - \mathcal{K}^{(n-3)}$.

□

For an arbitrary simplicial complex $\mathcal{K}$, the argument of the proof fails for the points $p \in \mathcal{K}^{(n-3)}$ for the following reason. For simplicity, assume $\mathcal{K}$ is 3-dimensional. Then, a neighborhood of

$p \in \mathcal{K}$ can be a cone over any arbitrary 2-simplicial complex $\mathcal{K}'$. For instance, a neighborhood of $p \in \mathcal{K}$ can be a cone over T^2 which is the quotient topological space

$$\mathrm{Cone}(T^2) := \frac{[0,1] \times T^2}{(0,x) \sim (0,y) \qquad \forall\, x, y \in T^2}.$$

On a 3-manifold, however, a neighborhood of any point is a cone over S^2 (which is homeomorphic to a ball).

Theorem 3.7. *If a topological space X has the structure of a simplicial complex $\mathcal{K}$, then*

$$H_k^{\mathrm{simp}}(\mathcal{K}, \mathcal{R}) \cong H_k^{\mathrm{sing}}(X, \mathcal{R}) \qquad \forall\, k \geq 0.$$

In particular, simplicial homology is independent of the choice the simplicial complex structure on a topological space.

For instance, the last statement implies that any choice of a triangulation on the 2-torus yields the same homology groups as in Example 2.11. We will prove an extended version of Theorem 3.7 in Section **??**.

4 Homotopy invariance

In this section, we prove that homotopy equivalent topological spaces have isomorphic singular homologies. For instance, since every n-simplex is homotopy equivalent to a point, we conclude (2.6).

For every continuous map $f\colon X \longrightarrow Y$ and each $n \geq 0$, composition with f defines a map

$$f_\#\colon C_n^{\mathrm{sing}}(X, \mathcal{R}) \longrightarrow C_n^{\mathrm{sing}}(Y, \mathcal{R})$$

that commutes with boundary operators ∂ on the source and the target, inducing a push-forward homomorphism $f_*\colon H_n(X; \mathcal{R}) \longrightarrow H_n(X, \mathcal{R})$. Then, with a little bit of homological algebra, we show that f_* is an isomorphism whenever f is a homotopy equivalence.

If $f\colon X \longrightarrow Y$ is continuous map and $\sigma\colon \Delta \longrightarrow X$ is a singular n-simplex, then

$$f_\#(\sigma) := \sigma \circ f\colon \Delta \longrightarrow Y$$

is a singular n-simplex in Y. Extending this map linearly via

$$f_\#\Big(\sum_{i \in \mathcal{I}} a_i \sigma_i\Big) = \sum_{i \in \mathcal{I}} a_i f_\#(\sigma_i)$$

defines a map

$$f_\#\colon C_n^{\mathrm{sing}}(X, \mathcal{R}) \longrightarrow C_n^{\mathrm{sing}}(Y, \mathcal{R}) \qquad \forall\, n \geq 0.$$

Since the definition of ∂ involves boundary components of Δ and their boundary orientations (which is independent of the choice of σ), and since restriction to $\partial\Delta$ and composition with f commute, we have $f_\#\partial = \partial f_\#$ (the first partial takes place on the singular chain complex of

X and the second one takes place on the singular chain complex of Y). In other words, the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X,\mathcal{R}) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_n(X,\mathcal{R}) \arrow[r, "\partial"] \arrow[d, "f_\#"] & C_{n-1}(X,\mathcal{R}) \arrow[r, "\partial"] \arrow[d, "f_\#"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y,\mathcal{R}) \arrow[r, "\partial"] & C_n(Y,\mathcal{R}) \arrow[r, "\partial"] & C_{n-1}(Y,\mathcal{R}) \arrow[r, "\partial"] & \cdots
\end{tikzcd} \tag{4.1}$$

In the language of homological algebra, a commutative diagram as above is called a chain map $f_\#\colon \big(C_\bullet(X,\mathcal{R});\partial\big) \longrightarrow \big(C_\bullet(Y,\mathcal{R});\partial\big)$ between the chain complexes $\big(C_\bullet(X,\mathcal{R});\partial\big)$ and $\big(C_\bullet(Y,\mathcal{R});\partial\big)$. For every chain map as above, the commutativity relation $f_\#\partial = \partial f_\#$ implies that cycles are mapped to cycles and boundaries are mapped to boundaries. Therefore, $f_\#$ induces homomorphisms f_* between their quotient spaces which are the homology groups of the two complexes. It is easy to check that $\mathrm{id}_* = \mathrm{id}$ and $(f\circ g)_* = f_*\circ g_*$.

Often, a lot of information about $f_\#$ is lost in passing to homology. Therefore, a natural question to ask is: *when two different chain maps $f_\#$ and $g_\#$ induce the same homomorphisms $f_* = g_*$ between the homology groups of the two complexes?*

For abstract chain maps

$$f_\#,\ g_\#\colon (C_\bullet,\partial) \longrightarrow (C'_\bullet,\partial'),$$

between chain complexes $(C_\bullet,\partial)$ and $(C'_\bullet,\partial')$, a chain homotopy is a (sequence of) degree increasing map $h_\#$ as shown in the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & C_{n+1}, \arrow[r, "\partial"] \arrow[d, "g_\#"', "f_\#"] & C_n \arrow[r, "\partial"] \arrow[d, "g_\#"', "f_\#"] \arrow[dl, "h_\#"'] & C_{n-1} \arrow[r] \arrow[d, "g_\#"', "f_\#"] \arrow[dl, "h_\#"'] & \cdots \arrow[d] \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'"] & C'_n \arrow[r, "\partial'"] & C'_{n-1} \arrow[r] & \cdots ,
\end{tikzcd}$$

such that

$$\partial'\ h_\# + h_\#\ \partial = f_\# - g_\# \qquad \forall\ n\in\mathbb{Z}.$$

Lemma 4.1. *Chain homotopic chains maps $f_\#$ and $g_\#$ induce the same maps between homology groups.*

Warning. The last diagram is not commutative!

Back to the example of the push-forward chain maps (4.1) induced by a continuous functions $f\colon X\longrightarrow Y$ and $g\colon X\longrightarrow Y$, recall that we say $f,g\colon X\longrightarrow Y$ are homotopic if there exists a continuous function

$$h\colon [0,1]\times X \longrightarrow X$$

such that $h|_{\{0\}\times X} = g$ and $h|_{\{1\}\times X} = f$.

Lemma 4.2. *A topological homotopy h as above induces an algebraic homotopy between*

$$f_\#,\ g_\#\colon (C_\bullet(\mathcal{K},\mathcal{R}),\partial) \longrightarrow (C_\bullet(Y,\mathcal{R}),\partial),$$

Proof. For each singular n-simplex $\sigma\colon \Delta \longrightarrow X$, and any natural decomposition of $[0,1]\times\Delta$ into a union of oriented (with product orientation) $(n+1)$-simplices, the map

$$h\circ(\mathrm{id}\times\sigma)\colon [0,1]\times\Delta \longrightarrow Y$$

can be seen as singular $(n+1)$-chain in $C_{n+1}(Y,\mathcal{R})$. This linearly extends to all singular n-chains,

$$h_{\#}\colon C_n(X,\mathcal{R}) \longrightarrow C_{n+1}(Y,\mathcal{R}).$$

An oriented comparison of the boundary components shows that $h_{\#}$ is a chain homotopy between $f_{\#}$ and $g_{\#}$. $\square$

Corollary 4.3. *If two continuous maps $f,g\colon M \longrightarrow M'$ are homotopic, their induced homomorphisms between the homology groups of X and Y*

$$f_*,g_*\colon H_n^{\mathrm{sing}}(X,\mathcal{R}) \longrightarrow H_n^{\mathrm{sing}}(Y,\mathcal{R}) \qquad \forall\, n\geq 0 \tag{4.2}$$

are the same.

We say topological spaces X and Y are homotopic if there are continuous maps

$$f\colon X\longrightarrow Y \quad \text{and} \quad g\colon Y\longrightarrow X$$

such that $g\circ f$ is homotopic to id_X and $f\circ g$ is homotopic to id_Y.

Exercise 4.4. (Theorem) Show that homotopic topological spaces have identical singular homology groups.

Exercise 4.5. Prove that the singular homology of a point is concentrated at degree 0. Use the result in Exercise 4.4 to prove that if B is a ball in $\mathbb{R}^n$, then

$$H_k^{\mathrm{sing}}(B,\mathcal{R}) = \begin{cases} \mathcal{R} & \text{if} \quad k=0; \\ 0 & \text{otherwise.}\end{cases}$$

Exercise 4.6. Show that the simplicial complex X obtained by identifying the oriented edges $\Delta_{[v_0,v_1]}\sim\Delta_{[v_1,v_3]}$ and $\Delta_{[v_0,v_2]}\sim\Delta_{[v_2,v_3]}$ in the 3-simplex $\Delta_{[v_0,v_1,v_2,v_3]}$ deformation retracts to a Klein bottle. Use the result of Exercise 4.4 and Theorem 3.7 to calculate the integral homology groups of the Klein bottle.

5 Relative homology groups

Given a topological space X and a subspace $Y\subset X$, we can try to decompose the homology groups of X as a sum of the homology groups supported in Y and their complement. A precise formulation of the latter involves the quotient space

$$X/Y := \frac{X}{x\sim y \quad \forall\, x,y\in Y}.$$

A different idea for collapsing Y to a point is to consider the union of X and the cone over Y,

$$\mathrm{Cone}(Y) = \frac{Y\times[0,1]}{(x,0)\sim(x,1) \quad \forall\, x,y\in Y},$$

along $Y \subset X$ and $Y \cong Y \times \{1\} \subset \mathrm{Cone}(Y)$ (with the quotient topology on $X \cup_Y \mathrm{Cone}(Y)$). Without further assumptions on $Y \subset X$, both of these spaces can be quite complicated; for example, think about $\mathbb{Q} \subset \mathbb{R}$.
Instead of working with singular cycles in X/Y or $X \cup_Y \mathrm{Cone}(Y)$, it will more convenient to work with cycles in X with boundary in Y which leads to the concept of relative homology groups $H_n(X, Y, \mathcal{R})$. One can use relative homology to prove

$$H_n(X \cup_Y \mathrm{Cone}(Y), \mathcal{R}) \cong H_n(X, Y, \mathcal{R}) \qquad \forall\; n > 0. \tag{5.1}$$

Furthermore, under a reasonable condition on (X, Y), we will show that

$$H_n(X, Y, \mathcal{R}) \cong H_n(X/Y, \mathcal{R}) \qquad \forall\; n > 0.$$

Given a topological space X and a subspace $Y \subset X$, let

$$C_n(X, Y, \mathcal{R}) \coloneqq \frac{C_n(X, \mathcal{R})}{C_n(Y, \mathcal{R})}.$$

Since ∂ maps the subgroup $C_n(Y, \mathcal{R})$ to itself, it induces a similarly denoted map

$$\partial \colon C_n(X, Y, \mathcal{R}) \longrightarrow C_{n-1}(X, Y, \mathcal{R}) \qquad \forall\; n \in \mathbb{Z}$$

satisfying $\partial^2 = 0$ (because $\partial^2 = 0$ on $C_n(X, \mathcal{R})$). The relative singular homology groups $H_n^{\mathrm{sing}}(X, Y, \mathcal{R})$ are the homology groups of the chain complex

$$\big(C_\bullet(X, Y, \mathcal{R}), \partial\big) \coloneqq \Big\{ \cdots \longrightarrow C_{n+1}(X, Y, \mathcal{R}) \xrightarrow{\partial} C_n(X, Y, \mathcal{R}) \xrightarrow{\partial} C_{n-1}(X, Y, \mathcal{R}) \longrightarrow \cdots \Big\}$$

Remark 5.1. The definition above also applies to the case where X is a simplicial complex, $Y \subset X$ is a sub-simplicial complex, and $(C_\bullet(X, \mathcal{R}), \partial)$ and $(C_\bullet(Y, \mathcal{R}), \partial)$ are the simplicial chain complexes of X and Y, leading to the relative simplicial homology groups $H_n^{\mathrm{sing}}(X, Y, \mathcal{R})$ instead.

By definition, every homology group in $H_n^{\mathrm{sing}}(X, Y, \mathcal{R})$ is represented by an n-cycle in X such that $\beta = \partial\alpha$ is an $(n-1)$-chain in $C_{n-1}(Y, \mathcal{R})$. Furthermore, since $\partial^2 = 0$, we have $\partial\beta = 0$; i.e., β is an $(n-1)$-cycle in Y defining a homology class $[\beta] \in H_{n-1}^{\mathrm{sing}}(Y, \mathcal{R})$. We will see that the $[\alpha] \longrightarrow [\beta]$ is well-defined a gives a so-called connecting homomorphism $H_n^{\mathrm{sing}}(X, Y, \mathcal{R}) \longrightarrow H_{n-1}^{\mathrm{sing}}(Y, \mathcal{R})$.

Exercise 5.2. Prove that if X and Y are both path connected, then $H_0^{\mathrm{sing}}(X, Y, \mathcal{R}) = 0$,

The chain complexes $(C_\bullet(X, \mathcal{R}), \partial)$, $(C_\bullet(Y, \mathcal{R}), \partial)$, and $(C_\bullet(X, Y, \mathcal{R}), \partial)$ fit into a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(Y, \mathcal{R}) \arrow[r, "\partial"] \arrow[d, "\iota"] & C_n(Y, \mathcal{R}) \arrow[r, "\partial"] \arrow[d, "\iota"] & C_{n-1}(Y, \mathcal{R}) \arrow[r, "\partial"] \arrow[d, "\iota"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(X, \mathcal{R}) \arrow[r, "\partial"] \arrow[d, "\pi"] & C_n(X, \mathcal{R}) \arrow[r, "\partial"] \arrow[d, "\pi"] & C_{n-1}(X, \mathcal{R}) \arrow[r, "\partial"] \arrow[d, "\pi"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(X, Y, \mathcal{R}) \arrow[r, "\partial"] & C_n(X, \mathcal{R}) \arrow[r, "\partial"] & C_{n-1}(X, \mathcal{R}) \arrow[r, "\partial"] & \cdots
\end{tikzcd} \tag{5.2}$$

where the vertical maps ι and π are the inclusion and projection maps, respectively. For each n, the vertical sequence

$$0 \longrightarrow C_n(Y,\mathcal{R}) \xrightarrow{\iota} C_n(X,\mathcal{R}) \xrightarrow{\iota} C_n(X,Y,\mathcal{R}) \longrightarrow 0$$

is exact meaning that $\mathrm{Ker}(\pi) = \mathrm{Im}(\iota)$. More generally, an abstract chain complex

$$\cdots \longrightarrow C_{n+1} \xrightarrow{f_{n+1}} C_n \xrightarrow{f_n} C_{n-1} \longrightarrow \cdots$$

is said to be exact if $\mathrm{Ker}(f_n) = \mathrm{Im}(f_{n+1})$ for all $n \in \mathbb{Z}$. Exact chain complexes are those with trivial homology groups. The commutative diagram can be seen as a short exact sequence of chain-complexes

$$0 \longrightarrow \big(C_n(Y,\mathcal{R}),\partial\big) \xrightarrow{\iota} \big(C_n(X,\mathcal{R}),\partial\big) \xrightarrow{\pi} \big(C_n(X,Y,\mathcal{R}),\partial\big) \longrightarrow 0.$$

Proposition 5.3. *A short exact sequence of chain complexes*

$$0 \longrightarrow \big(A_\bullet,\partial\big) \xrightarrow{\iota} \big(B_\bullet,\partial\big) \xrightarrow{\pi} \big(C_\bullet,\partial\big) \longrightarrow 0.$$

results in a long-exact sequence of homology groups

$$\cdots \longrightarrow H_n(A_\bullet) \xrightarrow{\iota_\bullet} H_n(B_\bullet) \xrightarrow{\pi_*} H_n(C_\bullet) \xrightarrow{\delta} H_{n-1}(A_\bullet) \longrightarrow \cdots$$

where ι_* and π_* are the homomorphisms induced by ι and π.

Corollary 5.4. *If Y is a topological subspace of X, there is a long-exact sequence of singular homology groups*

$$\cdots \longrightarrow H_n^{\mathrm{sing}}(Y,\mathcal{R}) \xrightarrow{\iota_*} H_n^{\mathrm{sing}}(X,\mathcal{R}) \xrightarrow{\pi_*} H_n^{\mathrm{sing}}(X,Y,\mathcal{R}) \xrightarrow{\delta} H_{n-1}^{\mathrm{sing}}(Y,\mathcal{R}) \longrightarrow \cdots . \tag{5.3}$$

The same holds for simplicial homology if X is a simplicial complex and Y is a sub-simplicial complex.

Note that if both X and Y are path-connected, then $H_0^{\mathrm{sing}}(Y,\mathcal{R}) \cong H_0^{\mathrm{sing}}(X,\mathcal{R}) \cong \mathcal{R}$ and $H_0^{\mathrm{sing}}(X,Y,\mathcal{R}) = 0$; thus, the long-exact sequence ends with

$$\cdots \longrightarrow H_1^{\mathrm{sing}}(Y,\mathcal{R}) \xrightarrow{\iota_*} H_1^{\mathrm{sing}}(X,\mathcal{R}) \xrightarrow{\pi_*} H_1^{\mathrm{sing}}(X,Y,\mathcal{R}) \longrightarrow 0$$

Example 5.5. Let us consider the case where $Y = \{x\} \subset X$ is a single point. By Exercise 4.5, we have $H_n^{\mathrm{sing}}(x,\mathcal{R}) = 0$ for all $n > 0$. Also, the map $H_0^{\mathrm{sing}}(x,\mathcal{R}) \xrightarrow{\iota_*} H_0^{\mathrm{sing}}(X,\mathcal{R})$ is injective. Therefore, the long-exact sequence of Corollary 5.4 reads

$$H_n^{\mathrm{sing}}(X,\mathcal{R}) \cong H_n^{\mathrm{sing}}(X,x,\mathcal{R}) \qquad \forall\ n > 0.$$

The main result of Section 4 also applies to relative homology groups in the following sense. Suppose $Y \subset X$ and $Y' \subset X'$ are subspaces. If $f\colon X \longrightarrow X'$ is a continuous map, we say f is a map of pairs (X,Y) and (X',Y'), and write $f\colon (X,Y) \longrightarrow (X',Y')$, if $f(Y) \subset Y'$ (i.e., f restricts to a map between Y and Y').

Proposition 5.6. *Every continuous map $f\colon (X,Y) \longrightarrow (X',Y')$ induces a homomorphism*

$$f_*\colon H_n^{\mathrm{sing}}(X,Y,\mathcal{R}) \longrightarrow H_n^{\mathrm{sing}}(X',Y',\mathcal{R}) \qquad \forall\ n \geq 0.$$

If two such maps $f,g\colon (X,Y) \longrightarrow (X',Y')$ are homotopic through maps of pairs $(X,Y) \longrightarrow (X',Y')$, then $f_* = g_*$.

Exercise 5.7. Check that the arguments of Section 4 readily generalize to the relative setting to prove Proposition 5.6.

Also, if we have a sequence of subspaces $Z \subset Y \subset X$, the argument leading to the long-exact sequence of Corollary 5.4 generalizes to the short exact sequence of chain complex

$$0 \longrightarrow \big(C_n(Y,Z,\mathcal{R}),\partial\big) \xrightarrow{\iota} \big(C_n(X,Z,\mathcal{R}),\partial\big) \xrightarrow{\pi} \big(C_n(X,Y,\mathcal{R}),\partial\big) \longrightarrow 0$$

and gives us the long-exact sequence of homology groups

$$\cdots \longrightarrow H_n^{\mathrm{sing}}(Y,Z,\mathcal{R}) \xrightarrow{\iota_*} H_n^{\mathrm{sing}}(X,Z,\mathcal{R}) \xrightarrow{\pi_*} H_n^{\mathrm{sing}}(X,Y,\mathcal{R}) \xrightarrow{\delta} H_{n-1}^{\mathrm{sing}}(Y,Z,\mathcal{R}) \longrightarrow \cdots . \tag{5.4}$$

The long-exact sequence of Corollary 5.4 is a special case of (5.4) for which $Z = \emptyset$. The statement above also hold for simplicial homology if $Z \subset Y \subset X$ is a sequence of simplicial complexes.

Theorem 5.8. *(Excision Theorem) Given subspaces $Z \subset Y \subset X$ such that the closure of Z is contained in the interior of Y, then the homomorphisms*

$$\iota_*\colon H_n^{\mathrm{sing}}(X-Z, Y-Z, \mathcal{R}) \longrightarrow H_n^{\mathrm{sing}}(X,Y,\mathcal{R}) \qquad \forall\ n \geq 0,$$

induced by the inclusion map of pairs $\iota\colon (X-Z, Y-Z) \longrightarrow (X,Y)$ are isomorphisms. Equivalently, for subspaces $A, B \subset X$ whose interiors cover X, the homomorphisms

$$\iota_*\colon H_n^{\mathrm{sing}}(A, A\cap B, \mathcal{R}) \longrightarrow H_n^{\mathrm{sing}}(X,B,\mathcal{R}) \qquad \forall\ n \geq 0,$$

induced by the inclusion map of pairs $\iota\colon (A, A\cap B) \longrightarrow (X,B)$ are isomorphisms.

The translation between the two statements is provided by setting $B = Y$ and $A = X - Z$.

Exercise 5.9. Show that the conditions $\mathrm{cl}(Z) \subset \mathrm{Int}(Y)$ and $X = \mathrm{Int}(A) \cup \mathrm{Int}(B)$ are equivalent.

Proof of Excision Theorem. In the course of the proof, and also in the proof of Mayer-Vietoris long-exact sequence below, we need to work with an (open) covering of X and singular simplices that have image in one of the components. For this purpose, we define a reduced chain complex whose singular simplices have the aforementioned property and compare it to the original chain complex in the following way.

Suppose $\mathcal{U} := \{U_\alpha\}_{\alpha\in\mathcal{I}}$ is a collection of subsets of X whose interiors form an open cover of X. For every $n \geq 0$, define $C_n^{\mathcal{U}}(X,\mathcal{R})$ to be the subgroup of $C_n(X,\mathcal{R})$ generated by singular n-simplices $\sigma\colon \Delta \longrightarrow X$ that have image in one of U_α for some $\alpha \in \mathcal{I}$. It is easy to check that $(C_\bullet^{\mathcal{U}}(X,\mathcal{R}),\partial)$ is a sub-chain complex of $(C_\bullet(X,\mathcal{R}),\partial)$.

Proposition 5.10. *The inclusion chain map $\iota\colon (C_\bullet^{\mathcal{U}}(X,\mathcal{R}),\partial) \longrightarrow (C_\bullet(X,\mathcal{R}),\partial)$ is a chain homotopy equivalence, that is, there is a reverse chain map $\rho\colon (C_\bullet(X,\mathcal{R}),\partial) \longrightarrow (C_\bullet^{\mathcal{U}}(X,\mathcal{R}),\partial)$ such that $\iota\circ\rho$ and $\rho\circ\iota$ are chain homotopic to the identity. Therefore, for every $n \geq 0$, the induced map ι_* between the homology groups $H_n^{\mathcal{U}}(X,\mathbb{R})$ of $(C_\bullet^{\mathcal{U}}(X,\mathcal{R}),\partial)$ and $H_n(X,\mathcal{R})$ is an isomorphism.*

Proof. Given a simplex Δ, the barycentric subdivision of Δ, denoted by $\mathfrak{b}(\Delta)$, is a standard way of dividing it into smaller sub-simplices of the same dimension by connecting the barycenters of their faces in a specific way. Formally, the barycentric subdivision is defined inductively in the following way. Starting with $n = 0$, the barycentric subdivision of a point is itself. Having the the barycentric subdivision defined for simplices of dimension less than n, for an n-simplex $\Delta = \Delta_{[v_0,\dots,v_n]}$, $\mathfrak{b}(\Delta)$ is the union of the cones formed by the center point

$$\frac{(v_0 + \cdots + v_n)}{n+1}$$

and $(n-1)$-simplices of $\mathfrak{b}(\partial\Delta)$. For instance, if $\Delta = [0,1]$, then $\mathfrak{b}(\Delta) = [0,1/2] \cup [1/2,1]$, and if $\Delta = \Delta_{[v_0,v_1,v_2]}$, then

$$\mathfrak{b}(\Delta) = \Delta_{[v_0,v_{01},v_{012}]} \cup \Delta_{[v_{01},v_1,v_{012}]} \cup \Delta_{[v_1,v_{12},v_{012}]} \cup \Delta_{[v_{12},v_2,v_{012}]} \cup \Delta_{[v_2,v_{20},v_{012}]} \cup \Delta_{[v_{20},v_0,v_{012}]},$$

where $v_{012} = (v_0+v_1+v_2)/3$ and $v_{ij} = v_{ji}$ is the mid point of Δ_{v_i,v_j}. If Δ is oriented, each simplex in $\mathfrak{b}(\Delta)$ will be considered with the same orientation. The actual choice of such a subdivision is not important in the following argument. The goal is to divide Δ into sufficiently small simplices (by reducing the diameter of each simplex) such that (after finitely many repetitions of this procedure) each one fits into at least one of U_α.

Exercise 5.11. Show that the barycentric subdivision of an n-simplex Δ consists of $(n+1)!$ n-simplices.

Given an arbitrary singular n-chain $\sigma = \sum_{i=1}^m (\sigma_i \colon \Delta_i \longrightarrow X)$, we can replace each Δ_i with its barycentric subdivision $\mathfrak{b}(\Delta_i) = \bigcup_{j=1}^{(n+1)!} \Delta_{ij}$ and define a similarly denoted chain map (it is clear that $\mathfrak{b}$ and ∂ commute)

$$\mathfrak{b}\colon C_n(X,\mathcal{R}) \longrightarrow C_n(X,\mathcal{R}), \quad \mathfrak{b}(\sigma) = \sum_{i=1}^{m} \sum_{j=1}^{(n+1)!} (\sigma_i|_{\Delta_{ij}} \colon \Delta_{ij} \longrightarrow X).$$

Lemma 5.12. *The map $\mathfrak{b}$ is chain homotopic to the identity map.*

Proof. For each n-simplex Δ, there is a systematic (inductive) way of decomposing $[0,1] \times \Delta$ into n-simplices (say by induction over n), denoted by $h(\Delta \times [0,1]) = \bigcup_i \Delta_i$, such that

- there is one $(n+1)$ simplex in $h(\Delta \times [0,1])$ that include $\{0\} \times \Delta$ as a codimension-1 face;
- for each simplex Δ' in the barycentric subdivision of Δ, there is one $(n+1)$ simplex in $h(\Delta \times [0,1])$ that include $\{1\} \times \Delta'$ as a codimension-1 face;
- the rest of the codimension-1 faces of Δ_i either belong to a canceling pair inside $\Delta \times [0,1]$ or belong to $[0,1] \times h(\partial\Delta)$.

Therefore, the items in 1-3 above give us the signed decomposition

$$\partial h(\Delta) = -\{0\} \times \Delta + \mathfrak{b}(\Delta) - h\partial(\Delta)$$

or equally

$$\partial h + h\partial = \mathfrak{b} - \mathrm{id}.$$

The linear extension of the map h to (the domain of the) singular n-chains gives us the desired chain homotopy equivalence between id and $\mathfrak{b}$. □

In general, if two chain maps $f_\#, g_\# \colon (C_\bullet, \partial) \longrightarrow (C'_\bullet, \partial')$ are chain homotopic, then iterations of these maps are chain homotopic as well. In the case of Lemma 5.12, the chain homotopy $h^{(k)}$ between id and $\mathfrak{b}^k = \mathfrak{b} \circ \cdots \circ \mathfrak{b}$ can be explicitly written as

$$h^{(k)} = h \circ \frac{\mathrm{id} - \mathfrak{b}^k}{\mathrm{id} - \mathfrak{b}} := h \sum_{a=0}^{k-1} \mathfrak{b}^a.$$

In order to define the map ρ in Proposition 5.10, given a singular n-chain σ, we find a non-negative integer $k \geq 0$ such that $\mathfrak{b}^k(\sigma)$ belongs to $C_n^{\mathcal{U}}(X, \mathcal{R})$. The existence of such k is provided by the Lemma below.

Lemma 5.13. *For every singular n-simplex $\sigma \colon \Delta \longrightarrow X$, there exists (a minimal) $k = k(\sigma) \geq 0$ such that $\mathfrak{b}^\ell(\sigma) \in C_n^{\mathcal{U}}(X, \mathcal{R})$ for all $\ell \geq k$.*

Proof. The pre-image $\{V_\alpha := \sigma^{-1}(U_\alpha^\circ)\}_{\alpha \in \mathcal{I}}$ of the open covering $\{U_\alpha^\circ = \mathrm{Int}(U_\alpha)\}_{\alpha \in cI}$ is an open covering of the compact topological space Δ (Thus, we may assume that $\mathcal{I}$ is finite; but that is not how we use compactness). Suppose that for all $k \geq 0$, there is an n-simplex Δ_k in $\mathfrak{b}^\ell(\Delta)$ that does not belong to any V_α. Let p_k by any point in Δ_k. Since Δ is compact, after passing to a subsequence, we have $\lim_{k \longrightarrow \infty} p_k = p$ for some $p \in \Delta$. We have $p \in V_\alpha$ for some $\alpha \in \mathcal{I}$. Then, a ball of radius ϵ around x is also contained in V_α. Since the diameter of the simplices Δ_k converges to 0, for k sufficiently large, Δ_k is contained in V_α. That is a contradiction! If $\mathfrak{b}^k(\sigma) \in C_n^{\mathcal{U}}(X, \mathcal{R})$ for some k, then it is clear that $\mathfrak{b}^\ell(\sigma) \in C_n^{\mathcal{U}}(X, \mathcal{R})$ for all $\ell \geq k$. ☐

In order to define the desired map

$$\rho \colon (C_\bullet(X, \mathcal{R}), \partial) \longrightarrow (C_\bullet^{\mathcal{U}}(X, \mathcal{R}), \partial)$$

we will use $\mathfrak{b}^k$ but we can not expect the same k to work for all singular simplices.

Exercise 5.14. For each singular simplex $\sigma \colon \Delta \longrightarrow X$ define

$$\begin{aligned} &H(\sigma) = h^{(k(\sigma))}(\sigma) \in C_{n+1}^{\mathcal{U}}(X, \mathcal{R}) \qquad \text{and} \\ &\rho(\sigma) = \mathfrak{b}^{(k(\sigma))}(\sigma) + h^{(k(\sigma))}(\partial\sigma) - h^{(k(\partial\sigma))}(\partial\sigma) \in C_n^{\mathcal{U}}(X, \mathcal{R}). \end{aligned}$$

Extend these definitions linearly over the chain complex to get

$$\rho \colon C_n(X, \mathcal{R}) \longrightarrow C_n^{\mathcal{U}}(X, \mathcal{R}) \qquad \text{and} \quad H \colon C_n(X, \mathcal{R}) \longrightarrow C_{n+1}^{\mathcal{U}}(X, \mathcal{R}).$$

for all $n \geq 0$. Prove $\partial H + H\partial = \mathrm{id} - \iota \circ \rho$ and $\mathrm{id} - \rho \circ \iota = 0$ to conclude that $\iota \circ \iota$ and $\rho \circ \iota$ are chain homotopic to the identity maps.

This finishes the proof of Proposition 5.10. We go back to the proof of Excision Theorem. We prove the second statement involving the open covering $X = A^\circ \cup B^\circ$.

The homology groups $H_\bullet(A, A \cap B, \mathcal{R}) \longrightarrow H_\bullet(X, B, \mathcal{R})$ come from the singular chain complexes

$$\left(\frac{C_\bullet(A, \mathcal{R})}{C_\bullet(A \cap B, \mathcal{R})}, \partial\right) \quad \text{and} \quad \left(\frac{C_\bullet(X, \mathcal{R})}{C_\bullet(B, \mathcal{R})}, \partial\right),$$

respectively. With $\mathcal{U} = \{A, B\}$, we show that the numerator of the right hand side can be replaced with $C_\bullet^{\mathcal{U}}(X, \mathcal{R})$ without affecting the homology groups. Then, since $C_\bullet^{\mathcal{U}}(X, \mathcal{R}) = C_\bullet(A, \mathcal{R}) + C_\bullet(B, \mathcal{R})$, we have

$$\left(\frac{C_\bullet(A, \mathcal{R})}{C_\bullet(A \cap B, \mathcal{R})}, \partial\right) = \left(\frac{C_\bullet(X, \mathcal{R})}{C_\bullet(B, \mathcal{R})}, \partial\right),$$

which finishes the proof of the Excision Theorem.

At the end of the preceding proof we had formulas $\partial H + H\partial = \mathrm{id} - \iota \circ \rho$ and $\rho \circ \iota = \mathrm{id}$. All the maps appearing in these formulas take chains in B to chains in B, so they induce quotient maps when we factor out chains in A. These quotient maps automatically satisfy the same two formulas, so the inclusion

$$\left(\frac{C_\bullet^{\mathcal{U}}(X,\mathcal{R})}{C_\bullet(B,\mathcal{R})}, \partial\right) \longrightarrow \left(\frac{C_\bullet(X,\mathcal{R})}{C_\bullet(B,\mathcal{R})}, \partial\right)$$

is a chain homotopy equivalence. □

Theorem 5.15. *Suppose $Z \subset X$ is a nonempty closed subspace that is a deformation retract of some neighborhood $Y \supset Z$ in X. Then, the quotient map $q\colon (X,Z) \longrightarrow (X/Z, x = Z/Z)$ induces isomorphisms $q_*\colon H_n^{\mathrm{sing}}(X,Z,\mathcal{R}) \longrightarrow H_n^{\mathrm{sing}}(X/Z,x,\mathcal{R})$ for all $n \geq 0$. In particular,*

$$H_n^{\mathrm{sing}}(X,Z,\mathcal{R}) \cong H_n^{\mathrm{sing}}(X/Z,\mathcal{R}) \qquad \forall\ n > 0.$$

Proof. The proof involves the long-exact sequence (5.4) and the first version of Excision Theorem. Since Z is a deformation retract of some neighborhood Y, the relative homology groups $H_n(Y,Z)$ are trivial for all $n \geq 0$. Therefore, it follows from the long-exact sequence

$$\cdots \longrightarrow H_n(Y,Z,\mathcal{R}) \xrightarrow{\iota_*} H_n(X,Z,\mathcal{R}) \xrightarrow{\pi_*} H_n(X,Y,\mathcal{R}) \xrightarrow{\delta} H_{n-1}(Y,Z,\mathcal{R}) \longrightarrow \cdots,$$

that

$$H_n(Y,Z,\mathcal{R}) \xrightarrow{\iota_*} H_n(X,Z,\mathcal{R})$$

is an isomorphism for all $n \geq 0$. By Excision Theorem, the map $H_n(X-Z,Y-Z) \longrightarrow H_n(X,Y)$ is also an isomorphism for all $n \geq 0$. The same statements hold for

$$H_n(X/Z,x) \longrightarrow H_n(X/Z,Y/Z) \quad \text{and} \quad H_n(X/Z-x,Y/Z-x) \longrightarrow H_n(X/Z,Y/Z).$$

Therefore, in the following (commutative) diagram

$$\begin{tikzcd}
H_n(X,Z) \arrow[r] \arrow[d, "q_*"] & H_n(X,Y) \arrow[d, "q_*"] & H_n(X-Z,Y-Z) \arrow[l] \arrow[d, "q_*"] \\
H_n(X/Z,x) \arrow[r] & H_n(X/Z,Y/Z) & H_n(X/Z-x,Y/Z-x) \arrow[l]
\end{tikzcd}$$

the horizontal maps are all isomorphisms. Away from Z, the retraction map $q\colon X-Z \longrightarrow X/Z-x$ is a homeomorphism. Thus, the q_* in the right column is an isomorphism. It follows that the other two columns are isomorphisms as well. Finally, for $n > 0$, we have $H_n(X/Z,x) \cong H_n(X/Z)$. □

Exercise 5.16. Prove (5.1).

Example 5.17. Theorem 5.15 is a strong tool for doing calculations. For instance, if X is an m-dimensional ball $B_m \subset \mathbb{R}^m$ and $Z = \partial X = S^{m-1}$, then X/Z is the m-sphere S^m. We conclude that

$$H_n^{\mathrm{sing}}(S^m,\mathbb{Z}) \cong H_n^{\mathrm{sing}}(B_m,S^{m-1}) \qquad \forall\ n > 0.$$

Furthermore, the long-exact sequence (5.3) reads

$$\cdots \longrightarrow H_n^{\text{sing}}(S^{m-1},\mathbb{Z}) \xrightarrow{\iota_*} H_n^{\text{sing}}(B_m,\mathbb{Z}) \xrightarrow{\pi_*} H_n^{\text{sing}}(B_m,S^{m-1},\mathbb{Z}) \xrightarrow{\delta} H_{n-1}^{\text{sing}}(S^{m-1},\mathbb{Z}) \longrightarrow \cdots .$$

Since $H_n^{\text{sing}}(B_m,\mathbb{Z})=0$ for $n>0$, we conclude that

$$H_n^{\text{sing}}(B_m,S^{m-1},\mathbb{Z}) \cong H_{n-1}^{\text{sing}}(S^{m-1},\mathbb{Z}) \qquad \forall\ n\geq 2.$$

For $n=1$ and $m>1$, we get the exact sequence

$$0 \longrightarrow H_1^{\text{sing}}(B_m,S^{m-1},\mathbb{Z}) \xrightarrow{\delta} H_0^{\text{sing}}(S^{m-1},\mathbb{Z})=\mathbb{Z} \longrightarrow H_0^{\text{sing}}(B_m,\mathbb{Z})=\mathbb{Z} \longrightarrow 0,$$

which implies $H_1^{\text{sing}}(B_m,S^{m-1},\mathbb{Z})=0$. For $n=1$ and $m=1$, we get the exact sequence

$$0 \longrightarrow H_1^{\text{sing}}(B_m,S^{m-1},\mathbb{Z}) \xrightarrow{\delta} H_0^{\text{sing}}(S^{m-1},\mathbb{Z})=\mathbb{Z}^2 \longrightarrow H_0^{\text{sing}}(B_m,\mathbb{Z})=\mathbb{Z} \longrightarrow 0,$$

which implies $H_1^{\text{sing}}(B_1,S^0,\mathbb{Z})=\mathbb{Z}$ (note that S^0 is a union of two points). These relations can be used to inductively prove

$$H_n^{\text{sing}}(S^m,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if} \quad n=0,m; \\ 0 & \text{otherwise.} \end{cases}$$

Here is another application of the ideas developed in this section which shows every continuous manifold has a well-defined dimension.

Theorem 5.18. *If nonempty open subsets $U\subset\mathbb{R}^n$ and $V\subset\mathbb{R}^m$ are homeomorphic, then $m=n$.*

Proof. Suppose $f\colon U\longrightarrow V$ is a homeomorphism. Let $x\in U$ and $y=f(x)\in V$. For sufficiently small $\epsilon>0$, $U-x$ deformation retracts to $U-B_\epsilon(x)$, where $B_\epsilon(x)$ is a ball of radius ϵ around x. Furthermore, $U/(U-B_{\epsilon(x)})\cong S^m$. Therefore, by Theorem 5.15, $H_k(U,U-x,\mathbb{Z})=H_k(S^n,\mathbb{Z})$ for all $k>0$. Similarly, $H_k(V,V-x,\mathbb{Z})=H_k(S^m,\mathbb{Z})$ for all $k>0$. Since $H_k(U,U-x,\mathbb{Z})\cong H_k(V,V-x,\mathbb{Z})$, we conclude that $m=n$. □

Exercise 5.19. Suppose X is the wedge sum $\vee_\alpha(X_\alpha,x_\alpha)$ of $\{X_\alpha\}$ at the base points $x_\alpha\in X_\alpha$. If a neighborhood of each x_α in X_α deformation retracts to x_α, prove that

$$H_n(X)\cong\bigoplus_\alpha H_n(X_\alpha) \qquad \forall\ n>0.$$

We now have the necessary tools to prove Theorem 3.7.

Proof. Fix an orientation on each simplex of X. There is a canonical chain map

$$(C_\bullet^{\text{simp}}(X,\mathcal{R}),\partial) \longrightarrow (C_\bullet^{\text{sing}}(X,\mathcal{R}),\partial)$$

which sends each oriented simplex Δ of X to its characteristic map $\sigma\colon\Delta\longrightarrow X$. This chain map induces homomorphisms

$$H_n^{\text{simp}}(X,\mathcal{R}) \longrightarrow H_n^{\text{sing}}(X,\mathcal{R}) \qquad \forall\ n\geq 0. \tag{5.5}$$

If $Y \subset X$ is a sub simplicial complex, the chain map above restricts to

$$(C_\bullet^{\rm simp}(Y,\mathcal{R}),\partial) \longrightarrow (C_\bullet^{\rm sing}(Y,\mathcal{R}),\partial),$$

descends to

$$(C_\bullet^{\rm simp}(X,Y,\mathcal{R}),\partial) \longrightarrow (C_\bullet^{\rm sing}(X,Y,\mathcal{R}),\partial),$$

and thus induces homomorphisms

$$H_n^{\rm simp}(X,Y,\mathcal{R}) \longrightarrow H_n^{\rm sing}(X,Y,\mathcal{R}) \qquad \forall\ n \geq 0. \tag{5.6}$$

For every $k \geq 0$, recall that $X^{(k)}$ denotes the k-skeleton of X consisting of k-dimensional or less simplices.

Lemma 5.20. *For every $k,n \geq 0$, the canonical homomorphism*

$$H_n^{\rm simp}(X^{(k)},X^{(k-1)},\mathcal{R}) \longrightarrow H_n^{\rm sing}(X^{(k)},X^{(k-1)},\mathcal{R})$$

is an isomorphism.

Proof. The case $n=0$ is easy; we assume $n>0$. If $Y \subset X$ is a sub-simplicial complex, a neighborhood of Y in X retracts to Y. Therefore, Theorem 5.15 applies to $(X^{(k)},X^{(k-1)})$ and

$$H_n^{\rm sing}(X^{(k)},X^{(k-1)}) \cong H_n^{\rm sing}(X^{(k)}/X^{(k-1)}) \qquad \forall\ n>0.$$

If $\{\Delta_\alpha\}_{\alpha\in\mathcal{I}}$ is the collection of k-simplices in X, the quotient space $X^{(k)}/X^{(k-1)}$ is the wedge space $\vee_\alpha(X_\alpha,x_\alpha)$ where $X_\alpha=\Delta_\alpha/\partial\Delta_\alpha$ and $x_\alpha=\partial\Delta_\alpha/\partial\Delta_\alpha$. Since each X_α is a k-dimensional sphere, by Exercise 5.19,

$$H_n^{\rm sing}(X^{(k)}/X^{(k-1)},\mathcal{R})=0 \qquad \forall n\neq 0,k$$

and

$$H_k^{\rm sing}(X^{(k)}/X^{(k-1)},\mathcal{R})=\mathcal{R}^{\mathcal{I}}.$$

For the simplicial homology, the only non-trivial term of the relative chain complex $C_\bullet^{\rm simp}(X^{(k)},X^{(k-1)})),\partial)$ is the degree k term

$$C_k^{\rm simp}(X^{(k)},X^{(k-1)},\mathcal{R})=\frac{C_k^{\rm simp}(X^{(k)},\mathcal{R})}{C_k^{\rm simp}(X^{(k-1)},\mathcal{R})}=C_k^{\rm simp}(X^{(k)},\mathcal{R})\cong\mathcal{R}^{\mathcal{I}}.$$

Therefore,

$$H_n^{\rm simp}(X^{(k)},X^{(k-1)},\mathcal{R})=0 \qquad \forall n\neq 0,k$$

and

$$H_k^{\rm ssimp}(X^{(k)},X^{(k-1)},\mathcal{R})=\mathcal{R}^{\mathcal{I}}.$$

We conclude that

$$H_n^{\rm simp}(X^{(k)},X^{(k-1)},\mathcal{R}) \longrightarrow H_n^{\rm sing}(X^{(k)},X^{(k-1)},\mathcal{R})$$

is an isomorphism. $\square$

Lemma 5.21 (Five-Lemma)**.** *In a commutative diagram of abelian groups*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d] & E \arrow[d] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd},$$

if the rows are exact, and the first, second, fourth, and fifth vertical maps are isomorphisms, then the middle one is an isomorphism as well.

Going back to the proof of Theorem 3.7, first, suppose X is finite-dimensional. Assume that

$$H_n^{\text{simp}}(X^{(k-1)}) \longrightarrow H_n^{\text{sing}}(X^{(k-1)})$$

is an isomorphism for all $n \geq 0$. Moving inductively from $k-1$ to k, we have a commutative diagram

$$\begin{tikzcd} H_{n+1}^{\text{simp}}(X^{(k)}, X^{(k-1)}) \arrow[r] \arrow[d] & H_n^{\text{simp}}(X^{(k-1)}) \arrow[r] \arrow[d] & H_n^{\text{simp}}(X^{(k)}) \arrow[r] \arrow[d] & H_n^{\text{simp}}(X^{(k)}, X^{(k-1)}) \arrow[r] \arrow[d] & H_{n-1}^{\text{simp}}(X^{(k-1)}) \arrow[d] \\ H_{n+1}^{\text{sing}}(X^{(k)}, X^{(k-1)}) \arrow[r] & H_n^{\text{sing}}(X^{(k-1)}) \arrow[r] & H_n^{\text{sing}}(X^{(k)}) \arrow[r] & H_{n+1}^{\text{sing}}(X^{(k)}, X^{(k-1)}) \arrow[r] & H_n^{\text{sing}}(X^{(k-1)}) \end{tikzcd}$$

where the rows are (some segment of) the simplicial and singular long-exact sequence (5.3) associated to the pair $(X^{(k)}, X^{(k-1)})$ and the vertical maps are the homomorphism in (5.6). By the induction assumption, the second and fifth vertical maps are isomorphisms. By Lemma 5.20, the first and fourth vertical maps are isomorphisms. It follows from the so-called Five-Lemma that the middle vertical map

$$H_n^{\text{simp}}(X^{(k)}) \longrightarrow H_n^{\text{sing}}(X^{(k)})$$

is an isomorphism as well, for all $n \geq 0$.

Next, we consider the case that X is not necessarily finite-dimensional. We prove that (5.5) is both injective and surjective. We make use of the following lemma.

Lemma 5.22. *For every singular chain σ in a simplicial complex X, there is a sufficiently large k such that σ is supported inside $X^{(k)}$.*

Proof. Since every singular chain σ is a finite sum of singular simplices, it is sufficient to assume that $\sigma\colon \Delta \longrightarrow X$ is a singular simplex. Suppose that $\sigma^{-1}(X - X^{(k)}) \neq \emptyset$ for infinitely many k. Then there is a sequence $\{x_k\}_{k=1}^{\infty}$ such that $x_k \in \text{image}(\sigma)$, x_k belongs to the interior of an n_k-simplex, and $\lim_{k \longrightarrow \infty} n_k = \infty$. By the definition of quotient topology, each $U_i = X - \bigcup_{j \neq i} x_j$ is an open subset of X. Since Δ is compact, finitely many of $\sigma^{-1}(U_i)$ cover Δ. That is a contradiction. $\square$

Going back to the proof of surjectivity and injectivity of (5.5), for each $k \geq 0$, the diagram

$$\begin{tikzcd} H_n^{\text{simp}}(X^{(k)}) \arrow[r] \arrow[d] & H_n^{\text{simp}}(X) \arrow[d] \\ H_n^{\text{sing}}(X^{(k)}) \arrow[r] & H_n^{\text{sing}}(X) \end{tikzcd}$$

commutes and the left column is an isomorphism. Given a homology class in $H_n^{\mathrm{sing}}(X)$,
(i) represent it by a singular n-cycle σ;
(ii) by lemma above, find k such that σ is supported in $X^{(k)}$ (i.e. it is in the image of the second row);
(iii) since the left column is an isomorphism, find a simplicial n-cycle σ' in $C_n^{\mathrm{simp}}(X)$ that has the same homology class as σ.
We conclude that the right column is surjective.

Next, suppose σ in $C_n^{\mathrm{simp}}(X)$ is a simplicial n-cycle which is trivial in singular homology.
(i) Find a singular $(n+1)$-chain σ' such that $\partial\sigma' = \sigma$;
(ii) by lemma above, there is k such that σ' is supported in $X^{(k)}$; therefore σ has trivial singular homology in $X^{(k)}$;
(iii) since the left column is an isomorphism, find a simplicial $(n+1)$-chain σ'' in $C_{n+1}^{\mathrm{simp}}(X)$ that $\partial\sigma'' = \sigma$.
We conclude that σ has trivial homology class in $H_n^{\mathrm{simp}}(X)$; i.e. the right column is injective. $\square$

The following generalization of Theorem 3.7 follows from Theorem 3.7, the long-exact sequence (5.3), and the Five-Lemma above.

Theorem 5.23. *If X is a simplicial complex and $Y \subset X$ is a simplicial subcomplex then*

$$H_n^{\mathrm{simp}}(X, Y, \mathcal{R}) \cong H_n^{\mathrm{sing}}(X, Y, \mathcal{R}) \qquad \forall\ n \in \mathbb{Z}.$$

We finish this section with another corollary of Proposition 5.10 that is useful both for calculations and inductive proof of some other results.

Theorem 5.24 (Mayer-Vietoris)**.** *Suppose that $A, B \subset X$ are subspaces whose interiors cover X. There exists a long-exact sequence of (singular or simpicial) homology groups*

$$\cdots H_n(A\cap B) \longrightarrow H_n(A) \oplus H_n(B) \longrightarrow H_n(X) \longrightarrow H_{n-1}(A\cap B) \cdots\ ,$$

where $H_n(A) \oplus H_n(B) \longrightarrow H_n(X)$ is the sum of canonical homomorphisms $H_n(A), H_n(B) \longrightarrow H_n(X)$ and $H_n(A\cap B) \longrightarrow H_n(A) \oplus H_n(B)$ is direct sum of the canonical homomorphism $H_n(A\cap B) \longrightarrow H_n(A)$ and the negative of the canonical homomorphism $H_n(A\cap B) \longrightarrow H_n(A)$.

Proof. For the covering $\mathcal{U} = \{A, B\}$ of X, we have a short exact sequence of singular chain complexes

$$0 \longrightarrow \big(C_\bullet(A\cap B), \partial\big) \longrightarrow \big(C_\bullet(A), \partial\big) \oplus \big(C_\bullet(B), \partial\big) \longrightarrow \big(C_\bullet^{\mathcal{U}}(X), \partial\big) \longrightarrow 0\ ,$$

where $C_n(A) \oplus C_n(B) \longrightarrow C_n^{\mathcal{U}}(X)$ is the summation map and $C_n(A\cap B) \longrightarrow C_n(A) \oplus C_n(B)$ is direct sum of the inclusion into A and the negative of the inclusion into B. By Proposition 5.10, $\big(C_\bullet^{\mathcal{U}}(X), \partial\big)$ and $\big(C_\bullet(X), \partial\big)$ have identical homology groups. Thus, the long-exact sequence of the short exact sequence above yields the desired result. $\square$

Example 5.25. Consider the standard covering $S^2 = A \cup B$ with two open disks. The annulus $A \cap B \cong S^1 \times [0,1]$ deformation retracts to S^1 so it has the same homology groups as S^1. Both A and B deformation retract to a point so their homology is concentrated at degree 0. For this decomposition, the Mayer-Vietoris sequence

$$\begin{aligned}
0 \longrightarrow\, & H_2(S^1, \mathbb{Z}) \longrightarrow H_2(A, \mathbb{Z}) \oplus H_2(B, \mathbb{Z}) \longrightarrow H_2(S^2, \mathbb{Z}) \longrightarrow \\
& H_1(S^1, \mathbb{Z}) \longrightarrow H_1(A, \mathbb{Z}) \oplus H_1(B, \mathbb{Z}) \longrightarrow H_1(S^2, \mathbb{Z}) \longrightarrow \\
& H_0(S^1, \mathbb{Z}) \xrightarrow{j_*} H_0(A, \mathbb{Z}) \oplus H_0(B, \mathbb{Z}) \longrightarrow H_0(S^2, \mathbb{Z}) \longrightarrow 0\ .
\end{aligned}$$

reads

$$\begin{aligned} 0 \longrightarrow 0 \longrightarrow 0 \oplus 0 \longrightarrow H_2(S^2, \mathbb{Z}) \longrightarrow& \\ \mathbb{Z} \longrightarrow 0 \oplus 0 \longrightarrow H_1(S^2, \mathbb{Z}) \longrightarrow 0& \\ \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_0(S^2, \mathbb{Z}) \longrightarrow 0& \ . \end{aligned}$$

Therefore, $H_2(S^2, \mathbb{Z}) = \mathbb{Z}$. Since $H_0(S^1, \mathbb{Z}) \xrightarrow{j_*} H_0(A, \mathbb{Z})$ is injective, we also conclude that $H_1(S^2, \mathbb{Z}) = 0$.

6 Degree of maps

In this section, we define the degree of a continuous map $f\colon X \longrightarrow Y$ between connected closed oriented manifolds X and Y of the same dimension. This is useful in applications such as in integrating differential forms and will also be used in defining a CW complex.

Lemma 6.1. *Suppose M is a connected oriented closed smooth n-manifold. Then,*

$$H_n(M, \mathbb{Z}) \cong \mathbb{Z}.$$

Proof. Fix a sufficiently refined triangulation on M to give it the structure of an n-dimensional simplicial complex such that each $(n-1)$-simplex is the common face of exactly two n-simplices. If $\{\Delta_\alpha\}_{\alpha\in\mathcal{I}}$ is the set of n-simplices oriented with the restriction of the chosen orientation on M, then, by the last line, a simplicial n-chain

$$\sum_{\alpha\in\mathcal{I}} a_\alpha \Delta_\alpha$$

is in the kernel of ∂ if and only if $a_\alpha = a_\beta$ whenever Δ_α and Δ_β share a codimension-1 face. Since M is connected, we conclude that

$$a_\alpha = a_\beta \qquad \forall\ \alpha, \beta \in \mathcal{I}.$$

Conversely, $\mathbb{Z} \cdot \sum_{\alpha\in\mathcal{I}} \Delta_\alpha \subset \mathrm{Ker}(\partial_n) = H_n(M, \mathbb{Z})$. Therefore, $H_n(M, \mathbb{Z}) \cong \mathbb{Z}$. □

Definition 6.2. If $f\colon M \longrightarrow N$ is a continuous map between two connected oriented closed smooth n-manifolds, the degree of f is the integer $\deg(f) \in \mathbb{Z}$ such that the homomorphism

$$f_*\colon H_n(M, \mathbb{Z}) \cong \mathbb{Z} \longrightarrow H_n(N, \mathbb{Z}) \cong \mathbb{Z}$$

is multiplication by $\deg(f)$. If $f\colon M \longrightarrow N$ is a continuous map between two oriented smooth n-manifolds, and

$$U \cap f^{-1}(f(U)) = \{x\} \tag{6.1}$$

for a sufficiently small neighborhood of x, then the local degree or multiplicity of f at $x \in M$ is the integer $\deg_x(f) \in \mathbb{Z}$ such that the homomorphism

$$\mathbb{Z} \cong H_n(M, M-x, \mathbb{Z}) \cong H_n(U, U-x, \mathbb{Z}) \xrightarrow{f_*} H_n(N, N-f(x), \mathbb{Z}) \cong \mathbb{Z}$$

is multiplication by $\deg_x(f)$.

The condition (6.1) is required for f to be a map of pairs $(U, U - x) \longrightarrow (N - N - y)$. The definition above naturally extend to case that M is closed and oriented but not necessarily connected. If $M_1, \ldots, M_k$ are the connected components of M, we define

$$\deg(f) = \sum_{i=1}^{k} \deg(f|_{M_i}).$$

Lemma 6.3. *If $f\colon M \longrightarrow N$ is a continuous map of degree d between two oriented closed smooth n-manifolds and N is connected, then*

$$\sum_{x \in f^{-1}(y)} \deg_x(f) = d \qquad \forall\ y \in N.$$

In particular, if $d \neq 0$, then f is surjective.

Proof. The case where $n = 0$ is trivial. We assume $n > 0$. Then,

$$H_n(N, N - y, \mathbb{Z}) \cong H_n(N, \mathbb{Z}) \cong \mathbb{Z} \qquad \forall\ y \in N,$$

and the first isomorphism corresponds to the inclusion $(N, \emptyset) \longrightarrow (N, N - y)$. We may assume M is connected. Then, the same holds for M. By Excision,

$$H_n(M, M - f^{-1}(y), \mathbb{Z}) \cong \bigoplus_{x \in f^{-1}(y)} H_n(M, M - x, \mathbb{Z}).$$

In the commutative diagram

$$\begin{tikzcd}
\mathbb{Z} \cong H_n(M, \mathbb{Z}) \arrow[r] \arrow[d] & H_n(N, \mathbb{Z}) \cong \mathbb{Z} \arrow[d] \\
H_n(M, M - f^{-1}(y), \mathbb{Z}) \cong \bigoplus_{x \in f^{-1}(y)} H_n(M, M - x, \mathbb{Z}) \arrow[r] & H_n(N, N - y, \mathbb{Z}) \cong \mathbb{Z}
\end{tikzcd},$$

the first row maps $1 \in \mathbb{Z}$ to $\deg(f) \in \mathbb{Z}$ and the right column maps 1 to 1. Therefore, their composition maps 1 to $\deg(f)$. Also, the left column maps 1 to

$$\bigoplus_{x \in f^{-1}(x)} 1 \in \mathbb{Z}^{|f^{-1}(y)|}$$

and the second row maps each $1 \in H_n(M, M - x, \mathbb{Z}) \cong \mathbb{Z}$ to $\deg_x(f) \in H_n(M, M - y, \mathbb{Z}) \cong \mathbb{Z}$. Therefore, the composition of the left column and second row maps 1 to $\sum_{x \in f^{-1}(y)} \deg_x(f)$. We conclude that

$$\sum_{x \in f^{-1}(y)} \deg_x(f) = \deg(f).$$

□

Example 6.4. Let $M = N = S^n$ with $n > 0$.

(1) The degree of the antipodal diffeomorphism $x \longrightarrow -x$ is $(-1)^{n-1}$.

(2) If $n = 1$ and $f\colon S^1 \subset \mathbb{C} \longrightarrow S^1 \subset \mathbb{C}$ is given by $z \longrightarrow z^k$ then $\deg(f) = k$. For $n > 1$ and every $k \in \mathbb{Z}$, there is a similarly defined map with $\deg(f) = k$.

(3) By definition, homotopic maps have the same degree. For continuous maps between S^n and S^n, the converse is also true. The last statement and the second statement of (2) imply that the n-th homotopy group of S^n is $\mathbb{Z}$; we will prove this later.

(4) If $f\colon S^n \to S^n$ has no fixed points then f is homotopic to the antipodal map and $\deg(f) = (-1)^{n-1}$. For if $f(x) \neq x$, then the line segment from $-x$ to $f(x)$ in $\mathbb{R}^{n+1}$ does not contain the origin. Therefore, for every $t \in [0,1]$, the map

$$f_t\colon S^n \longrightarrow S^n, \qquad x \longrightarrow \frac{-tx + (1-t)f(x)}{|-tx + (1-t)f(x)|}$$

is continuous and $\{f_t\}_{t\in[0,1]}$ gives a homotopy between the antipodal map and f.

(5) S^n has a nowhere zero continuous vector field iff n is odd. If ζ is such a vector field, then $\xi = \frac{\zeta}{|\zeta|}$ is a unit vector field orthogonal to the normal vector field and

$$f_\theta\colon S^n \longrightarrow S^n, \qquad x \longrightarrow \cos(\theta)x + \sin(\theta)\xi(x), \qquad \forall\, \theta \in [0,\pi],$$

is a homotopy between id and $-$id. Therefore, $(-1)^{n-1} = 1$ or n is odd. If $n = 2m-1$ is odd, then

$$\zeta(x_1,\ldots,x_{2m}) = (x_2, -x_1, \ldots, x_{2m}, -x_{2m-1})$$

is such a vector field.

(6) If n is even, $\mathbb{Z}_2$ is the only nontrivial group that acts freely on S^n. Every homeomorphism has degree $+1$ or -1 depending on whether it is orientation preserving or not. Therefore, if G acts on S^n, then deg defines a function $\deg\colon G \longrightarrow \mathbb{Z}_2$. By (4), if the action is free, then every non-trivial element of G is mapped to -1; i.e. $\mathrm{Ker}(G \longrightarrow \mathbb{Z}_2)$ is trivial.

7 Euler characteristic

Definition 7.1. Suppose X is a topological space with finite topology, meaning that $H_n(X,\mathbb{Z}) \neq 0$ for finitely many n and $H_n(X,\mathbb{Z})$ has finite rank for all n. Then, the Euler characteristic of X is the alternating sum

$$\chi(X) = \sum_n (-1)^n \text{ rank } H_n(X,\mathbb{Z}).$$

Proposition 7.2. *Suppose X is a finite simplicial complex, then*

$$\chi(X) = \sum_n (-1)^n \#\{n\text{-simplices of } X\}$$

When X is a triangulated surface, Proposition 7.2 gives the famous formula

$$\chi(X) = \#\text{vertices} + \#\text{triangles} - \#\text{edges}.$$

Proposition 7.2 is a corollary of the following Lemma.

Lemma 7.3. *Suppose $(C_\bullet, \partial)$ is a finite chain complex over $\mathbb{Z}$, meaning that $C_n \neq 0$ for finitely many n and C_n is a finite rank $\mathbb{Z}$-module for all n. Then,*

$$\sum_n (-1)^n \text{ rank } H_n(C_\bullet) = \sum_n (-1)^n \text{ rank } C_n. \tag{7.1}$$

Proof. If $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is a short-exact sequence of finite-rank $\mathbb{Z}$-modules then

$$\text{rank } B = \text{rank } A + \text{rank } C.$$

If $(C_\bullet, \partial)$ is a chain complex, for each n,

$$0 \longrightarrow \text{Im}(\partial_{n+1}) \longrightarrow \text{Ker}(\partial_n) \longrightarrow H_n(C) \longrightarrow 0$$

is a short-exact sequence. Therefore,

$$\begin{aligned}\sum_n (-1)^n \text{ rank } H_n(C_\bullet) &= \sum_n (-1)^n \left(\text{rank Ker}(\partial_n) - \text{rank Im}(\partial_{n+1})\right) \\ &= \sum_n (-1)^n \big(\text{rank Ker}(\partial_n) + \text{rank Im}(\partial_n)\big).\end{aligned}$$

For each n,

$$0 \longrightarrow \text{Ker}(\partial_n) \longrightarrow C_n \longrightarrow \text{Im}(\partial_n) \longrightarrow 0.$$

is also a short-exact sequence. Therefore,

$$\sum_n (-1)^n \text{ rank } C_n = \sum_n (-1)^n \big(\text{rank Ker}(\partial_n) + \text{rank Im}(\partial_n)\big).$$

We conclude (7.1). □

Example 7.4. We show that there is a smooth degree 1 map from the 2-torus T to the 2-sphere S^2 (see Example 8.3 for a different argument and its generalization). However, we can use euler characteristic to show that there is no holomorphic map of degree 1 between any complex torus T and S^2. In the reverse direction, it can be shown (by passing to the universal cover) that any continuous map from the 2-sphere to the 2-torus has degree zero.

Let $\gamma_1, \gamma_2 \colon S^1 \longrightarrow \mathbb{R}^3$ be two disjoint (oriented) embedded loops. Define

$$f_{\gamma_1,\gamma_2} \colon T \longrightarrow S^2 \subset \mathbb{R}^3, \qquad f(\theta_1, \theta_2) = \frac{\gamma_1(\theta) - \gamma_2(\theta)}{|\gamma_1(\theta) - \gamma_2(\theta)|} \qquad \forall, \theta_1, \theta_2 \in S^1.$$

If $\gamma_1 \cup \gamma_2$ is a Hopf link, it can be shown that f has degree 1 (Exercise: find the relation between $\deg(f)$ and the linking number of arbitrary disjoint oriented knots γ_1 and γ_2.)

If T is a complex torus and $f \colon T \longrightarrow S^2 = \mathbb{CP}^1$ is a holomorphic map, then f is a local diffeomorphism away from finitely many critical points $p_1, \ldots, p_k \in T$ and, for each p_i, there is a local holomorphic coordinate z_i around p_i and a local holomorphic coordinate around $f(p_i)$ such that $f(z_i) = z_i^{m_i}$. Here $m_i > 0$ is the local degree $\deg_{p_i} f$. Suppose $\mathcal{K}$ is a triangulation of S^2 such that every critical value $f(p_i)$ is a vertex of $\mathcal{K}$ and each 1 and 2-simplex has $\deg(f)$ disjoint lifts to T (such a triangulation exists). Let $f^{-1}(\mathcal{K})$ denote the lifted triangulation on T. Then,

$$\begin{aligned}\chi(S^2) &= \#\text{vertices of } \mathcal{K} + \#\text{triangles of } \mathcal{K} - \#\text{edges of } \mathcal{K}, \\ \chi(T) &= \#\text{vertices of } f^{-1}(\mathcal{K}) + \#\text{triangles of } f^{-1}(\mathcal{K}) - \#\text{edges of } f^{-1}(\mathcal{K}).\end{aligned}$$

We have,

$$\begin{aligned}\#\text{triangles of } f^{-1}(\mathcal{K}) &= \deg(f) \cdot \#\text{triangles of } \mathcal{K}, \\ \#\text{edges of } f^{-1}(\mathcal{K}) &= \deg(f) \cdot \#\text{edges of } \mathcal{K}, \\ \#\text{vertices of } f^{-1}(\mathcal{K}) &= \deg(f) \cdot \#\text{vertices of } \mathcal{K} - \sum_{p_i} (m_i - 1).\end{aligned}$$

Therefore,

$$\chi(T) = \mathrm{def}(f) \cdot \chi(S^2) - \sum_{p_i}(m_i - 1). \tag{7.2}$$

If $\mathrm{def}(f) = 1$, then $\sum_{p_i}(m_i - 1) = 2$. Therefore, we either have two critical points of order 2 or one critical point of order 3. In the second scenario, $f\colon T - \{p_1\} \longrightarrow S^2 - \{f(p_1)\}$ is a homeomorphism between once-punctured torus and once-punctured sphere. But these two spaces have different first homology groups. In the first scenario, we get a homeomorphism between twice-punctured torus and once or twice-punctured sphere. Both are impossible because $H_1(T^2 - \{p_1, p_2\}) = \mathbb{Z}^4$ while $H_1(S^2 - \{q_1\}) = 0$ and $H_1(S^2 - \{q_1, q_2\}) = \mathbb{Z}$.

The argument leading to (7.2) holds for any holomorphic map $f\colon \Sigma_1 \longrightarrow \Sigma_2$ between two Riemann surfaces and yields the RiemannHurwitz formula

$$\chi(\Sigma_1) = \mathrm{def}(f) \cdot \chi(\Sigma_2) - \sum_{p \in \mathrm{Crit}(f)} m_p,$$

where $\mathrm{Crit}(f)$ is the set of critical points of f and $m_p = \deg_p(f)$ is the local degree or multiplicity of f at p. As a simple corollary of RiemannHurwitz formula, there is no non-constant holomorphic map from a Riemann surface of lower genus to a Riemann surface of higher genus.

8 Cellular Homology

The concept of CW complex was introduced Whitehead to meet the needs of homotopy theory. This class of topological spaces is broader and has some better categorical properties than simplicial complexes, but still retains a combinatorial nature that allows for computation (often with a much smaller complex). For each $n \geq 0$, a n-cell is a n-dimensional open ball in $\mathbb{R}^n$; i.e.,

$$\{x \in \mathbb{R}^n : |x| < 1\}.$$

A 0-dimensional CW complex is a topological space with discrete topology. A k-dimensional CW complex is constructed, inductively, by gluing the boundaries of a number of k-cells to a $(k-1)$-dimensional CW complex $\mathcal{C}^{(k-1)}$. The topology of the resulting k-dimensional CW complex $\mathcal{C}^{(k)}$ is the quotient topology defined by these gluing maps. One can either stop this inductive process at a finite stage, setting $\mathcal{C} = \mathcal{C}^{(n)}$ for some $n < \infty$, or continue indefinitely, setting $\mathcal{C} = \bigcup_n \mathcal{C}^{(n)}$. In the latter case, $\mathcal{C}$ is given the weak topology: A set $A \subset \mathcal{C}$ is open (resp. closed) iff $A \cap \mathcal{C}^{(n)}$ is open (resp. closed) in $\mathcal{C}^{(n)}$ for all $n \geq 0$.

The terminology "CW"-complex refers to the following two properties satisfied by CW complexes: (1) Closure-finiteness: The closure of each cell meets only finitely many other cells; (2) Weak topology: A set is closed iff it meets the closure of each cell in a closed set. CW complexes are Hausdorff, locally contractible, and a neighborhood of each sub-complex A of a CW complex X deformation retracts onto A. We refer to the Appendix of [?] for a proof and complete discussion of these and other properties of CW complexes.

Since the interior of every k-simplex is a k-cell, every simplicial complex $\mathcal{K}$ is naturally a CW complex. The converse is not true, the gluing maps of the inductive construction above can be quite complicated. In conclusion, we have the following hierarchy:

$$\text{smooth manifolds} \subsetneq \text{simplicial complexes} \subsetneq \text{CW complexes} \subsetneq \text{topological spaces}.$$

For $n \geq 0$, let $C_n(\mathcal{C}, \mathcal{R})$ denote the free abelian group generated (over a commutative ring $\mathcal{R}$) by all n-cells in $\{e_{n,\alpha}\}_{\alpha \in \mathcal{I}_n}$ (together with a choice of orientation on each n-cell); therefore,

$$C_n(\mathcal{C}, \mathcal{R}) = \bigoplus_{\alpha \in \mathcal{I}_n} \mathcal{R} \cdot e_{n,\alpha} \cong \mathcal{R}^{\mathcal{I}_n}.$$

For each oriented n-cell $e_{n,\alpha}$, the boundary $(n-1)$-chain $\partial e_{n,\alpha} \in C_{n-1}(\mathcal{C}, \mathcal{R})$ is defined in the following way.

Every n-cell $e_{n,\alpha}$ comes with a gluing map

$$\varphi_\alpha \colon \partial^{\mathrm{top}} e_{n,\alpha} = S^{n-1} \longrightarrow \mathcal{C}^{(n-1)}, \tag{8.1}$$

where $\partial^{top} e_{n,\alpha} \cong S^{n-1}$ is the topological boundary[3] of $e_{n,\alpha}$ (with the boundary orientation). Let $\mathcal{C}^{(n-1)}/\mathcal{C}^{(n-2)}$ denote the topological space obtained by collapsing $\mathcal{C}^{(n-2)}$ into a point and $\pi \colon \mathcal{C}^{(n-1)} \longrightarrow \mathcal{C}^{(n-1)}/\mathcal{C}^{(n-2)}$ denote the corresponding projection map. The quotient space $\mathcal{C}^{(n-1)}/\mathcal{C}^{(n-2)}$ is a bucket/wedge of (oriented) $(n-1)$-spheres attached to each other at a single point. For each $\beta \in \mathcal{I}_{n-1}$, by further collapsing all other $(n-1)$-spheres $[e_{n-1;\gamma}]$ in $\mathcal{C}^{(n-1)}/\mathcal{C}^{(n-2)}$, with $\gamma \in \mathcal{I}_{n-1} - \{\beta\}$, and with notation as in (8.1), we obtain a continuous map between two oriented $(n-1)$-spheres

$$\varphi_{\alpha,\beta} \colon \partial^{\mathrm{top}} e_{n,\alpha} = S^{n-1} \xrightarrow{\varphi_\alpha} \mathcal{C}^{(n-1)} \xrightarrow{\pi} \mathcal{C}^{(n-1)}/\mathcal{C}^{(n-2)} \xrightarrow{\pi_\beta} [e_{n-1,\beta}] \cong S^{n-1}.$$

Let $d_{\alpha,\beta} \in \mathbb{Z}$ denote the degree of this map. We define

$$\partial e_{n,\alpha} = \sum_{\beta \in \mathcal{I}_{n-1}} d_{\alpha,\beta} e_{n-1,\beta}.$$

The sum is finite due to the Closure-finiteness property mentioned above. The boundary operator ∂ linearly extends to all n-chains in $C_n(\mathcal{C}, \mathcal{R})$:

$$(\text{or } \partial_n)\ \partial \colon C_n(\mathcal{C}, \mathcal{R}) \longrightarrow C_{n-1}(\mathcal{C}, \mathcal{R}). \tag{8.2}$$

Note that the definition of degrees $d_{\alpha,\beta}$ and thus ∂ makes use of singular homology (groups of sphere). Thus, the proof that $\partial^2 = 0$ would naturally involve statements about relative singular homology groups of different skeleta of $\mathcal{C}$. The cellular homology groups

$$H_n^{\mathrm{cell}}(\mathcal{C}, \mathcal{R}) := \frac{Z_n(\mathcal{C}, \mathcal{R}) := \mathrm{Ker}\big(\partial \colon C_n(\mathcal{C}, \mathcal{R}) \longrightarrow C_{n-1}(\mathcal{C}, \mathcal{R})\big)}{B_n(\mathcal{C}, \mathcal{R}) := \mathrm{Image}\big(\partial \colon C_{n+1}(\mathcal{C}, \mathcal{R}) \longrightarrow C_n(\mathcal{C}, \mathcal{R})\big)}, \qquad \forall\, n \geq 0,$$

are the homology groups of the chain complex $(C_\bullet(\mathcal{C}, \mathcal{R}), \partial)$.

Theorem 8.1. *If $\mathcal{C}$ is a CW complex, then (8.2) is a chain map and*

$$H_n^{\mathrm{cell}}(\mathcal{C}, \mathcal{R}) \cong H_n^{\mathrm{sing}}(\mathcal{C}, \mathcal{R}) \qquad \forall\, n \in \mathbb{Z}.$$

Proof. The proof consists of several steps and lemmas. We list these steps and sketch the proofs or leave some of the proofs to the reader. We denote the singular homology groups with $H_\bullet$ (with the superscript sing).

[3] We are writing $\partial^{\mathrm{top}} e$ instead of just ∂e to distinguish it from the chain map defined below.

(1) There is a natural identification

$$H_n(\mathcal{C}^{(n)}/\mathcal{C}^{(n-1)}, \mathcal{R}) = C_n(\mathcal{C}, \mathcal{R}) \qquad \forall n \geq 0. \tag{8.3}$$

For $n > 0$, this identification follows from the fact that
(i) $H_n(\mathcal{C}^{(n)}/\mathcal{C}^{(n-1)}, \mathcal{R}) = H_n(\mathcal{C}^{(n)}, \mathcal{C}^{(n-1)}, \mathcal{R})$,
(ii) $\mathcal{C}^{(n)}/\mathcal{C}^{(n-1)}$ is a wedge of n-spheres indexed by the set of n-cells of $\mathcal{C}$, and
(iii) $H_n(S^n, \mathcal{R}) = \mathcal{R}$.

For $n = 0$, the identification is the identity map!

(2) We can use the long-exact sequence

$$\cdots \longrightarrow H_k(\mathcal{C}^{n-1}) \longrightarrow H_k(\mathcal{C}^n) \longrightarrow H_k(\mathcal{C}^{(n)}, \mathcal{C}^{(n-1)}) \longrightarrow H_{k-1}(\mathcal{C}^{n-1}) \longrightarrow \cdots$$

to conclude that
(i) $H_k(\mathcal{C}^{(n)}, \mathcal{R}) = 0$ for $k > n$,
(ii) the inclusion $\iota\colon \mathcal{C}^{(n)} \longrightarrow \mathcal{C}$ induces isomorphisms $\iota_*\colon H_k(\mathcal{C}^{(n)}, \mathcal{R}) \longrightarrow H_k(\mathcal{C}, \mathcal{R})$ for $k < n$.

(3) In the diagram,

$$\begin{tikzcd}
 & H_n(\mathcal{C}^{(n-1)}) = 0 \arrow[d] & & \\
H_n(\mathcal{C}^{(n)}, \mathcal{C}^{(n-1)}) \arrow[r, "\delta_n"] & H_{n-1}(\mathcal{C}^{(n-1)}) \arrow[r] \arrow[d, "j_{n-1}"] & H_{n-1}(\mathcal{C}^{(n)}) \cong H_{n-1}(\mathcal{C}) \arrow[r] & H_{n-1}(\mathcal{C}^{(n)}, \mathcal{C}^{(n-1)}) = 0 \\
 & H_{n-1}(\mathcal{C}^{(n-1)}, \mathcal{C}^{(n-2)}) & &
\end{tikzcd}$$

the row is (part of) the long-exact sequence of the pair $(\mathcal{C}^{(n)}, \mathcal{C}^{(n-1)})$, the column is (part of) the long-exact sequence of the pair $(\mathcal{C}^{(n-1)}, \mathcal{C}^{(n-2)})$, the vanishing on top is due to Item (2)(i), the vanishing on right is due to Item (1), and the isomorphism $H_{n-1}(\mathcal{C}^{(n)}) \cong H_{n-1}(\mathcal{C})$ is due to Item (2)(ii).

(4) **Claim.** The map

$$j_{n-1} \circ \delta_n\colon C_n(\mathcal{C}, \mathcal{R}) \cong H_n(\mathcal{C}^{(n)}, \mathcal{C}^{(n-1)}) \longrightarrow H_n(\mathcal{C}^{(n-1)}, \mathcal{C}^{(n-2)}) \cong C_{n-1}(\mathcal{C}, \mathcal{R})$$

given by the identifications (8.3) and (1)(i) coincides with ∂_n in (8.2).

Proof. The claim follows from

$$H_{n-1}(\mathcal{C}^{(n-1)}, \mathcal{C}^{(n-2)}) = H_{n-1}(\mathcal{C}^{(n-1)}/\mathcal{C}^{(n-2)}),$$

when $n > 0$, and the definition of δ_n.

(5) From $j_{n-1} \circ \delta_n \equiv \partial_n$, we get

$$\partial_n \partial_{n+1} = j_{n-1} \circ \delta_n \circ j_n \circ \delta_{n+1}.$$

However, $\delta_n \circ j_n = 0$ because

$$H_n(\mathcal{C}^{(n)}) \xrightarrow{j_n} H_n(\mathcal{C}^{(n)}, \mathcal{C}^{(n-1)}) \xrightarrow{\delta_n} H_{n-1}(\mathcal{C}^{(n-1)}) \longrightarrow \cdots$$

is part of the long-exact sequence of the pair $(\mathcal{C}^{(n)}, \mathcal{C}^{(n-1)})$.

(6) The isomorphism

$$H_n^{\mathrm{cell}}(\mathcal{C},\mathcal{R}) \cong H_n^{\mathrm{sing}}(\mathcal{C},\mathcal{R})$$

follows from the union of the diagram in (3) and the similar one with $n+1$ instead of n in the following way. First, we redraw and complete the diagram in (3) in the following way:

$$\begin{tikzcd}
 & & & & 0 \\
 & 0 \arrow[d] & & H_n(\mathcal{C}) \arrow[ur] & \\
 & H_n(\mathcal{C}^{(n)}) \arrow[d, "j_n"] \arrow[urr, "i_n"] & & & \\
H_{n+1}(\mathcal{C}^{(n+1)}, \mathcal{C}^{(n)}) \arrow[ur, "\delta_{n+1}"] \arrow[r, "\partial_{n+1}"] & H_n(\mathcal{C}^{(n)}, \mathcal{C}^{(n-1)}) \arrow[r, "\partial_n"] \arrow[d, "\delta_n"] & H_{n-1}(\mathcal{C}^{(n-1)}, \mathcal{C}^{(n-2)}) & & \\
 & H_{n-1}(\mathcal{C}^{(n-1)}) \arrow[ur, "j_{n-1}"] & & & \\
0 \arrow[ur] & & & &
\end{tikzcd}$$

With notation as in the diagram above, we want to prove

$$H_n(\mathcal{C}) = \frac{\mathrm{Ker}(\partial_n)}{\mathrm{Im}(\partial_{n+1})}$$

The exactness of top slanted long-exact sequence implies

$$H_n(\mathcal{C}) = \frac{H_n(\mathcal{C}^{(n)})}{\mathrm{Im}(\delta_{n+1})}.$$

Since j_n is injective, $j_n \colon \mathrm{Im}(\delta_{n+1}) \longrightarrow \mathrm{Im}(\partial_{n+1})$ is an isomorphism and

$$\frac{H_n(\mathcal{C}^{(n)})}{\mathrm{Im}(\delta_{n+1})} \cong \frac{\mathrm{Im}(j_n)}{\mathrm{Im}(\partial_{n+1})}.$$

Similarly, since j_{n-1} is injective,

$$\delta_n \colon \mathrm{Ker}(\partial_n) \longrightarrow \mathrm{Im}(\delta_n)$$

is an isomorphism and

$$\frac{\mathrm{Im}(j_n)}{\mathrm{Im}(\partial_{n+1})} = \frac{\mathrm{Ker}(\delta_n)}{\mathrm{Im}(\partial_{n+1})} = \frac{\mathrm{Ker}(\partial_n)}{\mathrm{Im}(\partial_{n+1})}.$$

Therefore,

$$H_n(\mathcal{C}) = \frac{\mathrm{Ker}(\partial_n)}{\mathrm{Im}(\partial_{n+1})}$$

□

Example 8.2. For each $m \geq 0$, the real projective space $\mathbb{RP}^m$ has the structure of a CW complex $\mathcal{C}_{\text{std}}$ made of exactly one cell in each dimension $0 \leq k \leq m$. The gluing map of the k-th cell e_k to $\mathcal{C}_{k-1} = \mathbb{RP}^{k-1}$ is the $(2:1)$-covering map $S^{k-1} \longrightarrow \mathbb{RP}^{k-1}$. The cells can be oriented in a way that the boundary maps of the chain complex

$$0 \longrightarrow C_m(\mathcal{C}_{\text{std}}, \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0(\mathcal{C}_{\text{std}}, \mathbb{Z}) \cong \mathbb{Z} \longrightarrow 0$$

are equal to

$$\partial = (1 + (-1)^k)\colon C_k(\mathcal{C}_{\text{std}}, \mathbb{Z}) \cong \mathbb{Z} \longrightarrow C_{k-1}(\mathcal{C}_{\text{std}}, \mathbb{Z}) \cong \mathbb{Z}.$$

In other words, we have

$$\cdots \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \longrightarrow 0$$

We conclude that

$$H_i(\mathbb{RP}^m, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ \mathbb{Z} & \text{if } i = m \text{ and } m \text{ is odd}, \\ \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} & \text{if } 0 < i < m,\ i = \text{odd} \\ 0 & \text{otherwise}. \end{cases}$$

For each $m \geq 0$, the complex projective space $\mathbb{CP}^m$ has the structure of a CW complex $\mathcal{C}_{\text{std}}$ made of exactly one cell in each even dimension $0 \leq 2k \leq 2m$. The gluing map of the $2k$-th cell e_{2k} to $\mathcal{C}_{2k-1} = \mathcal{C}_{2k-2} = \mathbb{CP}^{k-1}$ is the projection map $S^{2k-1} \longrightarrow \mathbb{CP}^{k-1}$. The latter is a fiber bundle with S^1-fibers. It follows that the boundary maps in

$$0 \longrightarrow C_{2m}(\mathcal{C}_{\text{std}}, \mathbb{Z}) \cong \mathbb{Z} \xrightarrow{\partial} C_{2m-1}(\mathcal{C}_{\text{std}}, \mathbb{Z}) \cong 0 \xrightarrow{\partial} C_{2m-2}(\mathcal{C}_{\text{std}}, \mathbb{Z}) \cong \mathbb{Z} \cdots$$

are all trivial and

$$H_i(\mathbb{CP}^m, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 2k,\ 0 \leq k \leq m, \\ 0 & \text{otherwise}. \end{cases}$$

Example 8.3. Consider the standard CW complex for the torus consisting of one 0-cell, two (oriented) 1-cells a and b, and one 2-cell that is attached along the boundary to the word $aba^{-1}b^{-1}$. Thus the complement of the 2-cell is the wedge $S^1 \vee S^1$ of the two loops corresponding to a and b and $T = (S^1 \vee S^1) \cup_\varphi e_2$ where $\varphi\colon \partial^{\text{top}} e_2 \longrightarrow S^1 \vee S^1$ is the attaching map. It follows that the quotient space $T/S^1 \vee S^1$ has the CW structure $e_0 \cup_{\widetilde{\varphi}} e_2$ where e_0 is one 0-cell and $\widetilde{\varphi}$ is the constant map $\partial e_2 \longrightarrow e_0$. We observe that the quotient is S^2. The quotient map $q\colon T \longrightarrow S^2$ obtained above has degree 1 map because it is a homeomorphism on e_2. The argument above can be used to show that any connected oriented closed smooth n-manifold M admits a degree one map to S^n. Using Morse theory, one can show that M admits a CW complex structure with only one n-cell. Then $M \longrightarrow M/M^{(n-1)} \cong S^n$ is a degree 1 map. For $n \neq 4$, it is known that every closed connected C^0 n-manifold has such a CW complex structure with one n-cell, even if it does not admit a smooth structure. If $n = 4$, however, it is not even known if every closed four-manifold has a CW complex structure.

Lemma 8.4. *Given an abelian group G and $n \geq 1$, there exists a path-connected CW complex $\mathcal{C}$ with $H_n(\mathcal{C}, \mathbb{Z}) \cong G$ and $H_k(\mathcal{C}, \mathbb{Z}) \cong 0$ for all $k \neq 0, n$.*

Proof. The proof is constructive. The resulting CW complexes $\mathcal{C} = \mathcal{C}(G, n)$ are called Moore spaces.

If $G = \mathbb{Z}$, take $\mathcal{C} = S^n$. If $G = \mathbb{Z}$, start with $\mathcal{C}^{(n)} = S^n$ and attach an $(n+1)$-cell e_{n+1} via a C^0-map $\varphi\colon \partial^{\text{top}} e_{n+1} = S^n \longrightarrow S^n$ of degree d to build the desired $\mathcal{C}$. More generally, if G has a set of generators $\{x_\alpha\}_{\alpha\in\mathcal{I}}$, start with the wedge $\mathcal{C}^{(n)}$ of a set of n-spheres indexed by $\mathcal{I}$, and attach $(n+1)$-cells corresponding to the relations among the generators in the following way. The relations form the Kernel K of a map $F \longrightarrow G$, where F is the free abelian group generated by $\{x_\alpha\}_{\alpha\in\mathcal{I}}$. Thus K itself is a free abelian group generated by some set $\{y_\beta\}_{\beta\in\mathcal{I}'}$. Each y_β has a finite decomposition

$$y_\beta = \sum d_{\beta,\alpha} x_\alpha$$

where $d_{\beta,\alpha}$ are integers. The attaching homomorphisms

$$\varphi_\beta\colon \partial^{\text{top}} e_{n+1,\beta} \to \mathcal{C}^{(n)}$$

are dictated by the equations above. They can be constructed in the following way. For each β, choose a finite disjoint set $\{U_{\beta,\alpha}\}_{\alpha\colon d_{\beta,\alpha}\neq 0}$ of (open) balls in $\partial^{\text{top}} e_{n+1,\beta}$. Since $U_{\beta,\alpha}/\partial^{\text{top}} U_{\beta,\alpha} \cong S^n$, there is a continuous map

$$\varphi_{\beta,\alpha}\colon U_{\beta,\alpha} \longrightarrow S^n_\alpha \subset \mathcal{C}^{(n)}$$

such that (i) $\varphi_{\beta,\alpha}$ maps a collar neighborhood of $\partial^{\text{top}} U_{\beta,\alpha} \cong S^n$ to the attaching point x of the wedge of spheres in $\mathcal{C}^{(n)}$, and (ii) the induced map

$$U_{\beta,\alpha}/\partial^{\text{top}} U_{\beta,\alpha} \cong S^n \longrightarrow \mathcal{C}^{(n)}$$

is a degree $d_{\beta,\alpha}$ covering of $S^n_\alpha \subset \mathcal{C}^{(n)}$. By Property (i), there is C^0-map

$$\varphi_\beta\colon \partial^{\text{top}} e_{n+1,\beta} \to \mathcal{C}^{(n)}$$

whose restriction to each $U_{\beta,\alpha}$ is $\varphi_{\beta,\alpha}$ and is the constant function x on the complement

$$\partial^{\text{top}} e_{n+1,\beta} - \bigcup_\beta U_{\beta,\alpha}.$$

□

Remark 8.5. If X is a finite CW complex, the same argument as in the proof of Proposition 7.2 shows

$$\chi(X) = \sum_n (-1)^n \#\{n\text{-cells}\}.$$

For instance

$$\chi(\mathbb{RP}^n) = \begin{cases} 0 & \text{if } n = \text{odd} \\ 1 & \text{if } n = \text{even} \end{cases} \qquad \text{and} \qquad \chi(\mathbb{CP}^n) = n+1.$$

9 Cohomology

The homology groups $H_k(X, \mathbb{Z})$ of a topological space X are a bunch of abelian groups indexed by natural numbers $k \in \mathbb{N}$. A natural question is whether there is a product structure on

$$\bigoplus_{k\in\mathbb{N}} H_k(X, \mathbb{Z})$$

that would turn it into a ring? Let's consider the ideal case where X is a closed oriented smooth manifold and $[M]$ and $[N]$ are homology classes of degree m and n represented by oriented closed smooth submanifolds $M, N \subset X$. If M and N intersect transversally, then $M \cap N$ is also a closed oriented submanifold of dimension

$$\dim M \cap N = \dim M + \dim N - \dim X.$$

Thus, we can define

$$[M] \cap [N] := [M \cap N] \in H_{m+n-\dim X}(X, \mathbb{Z}).$$

It is easy to see that $[M] \cap [N]$ is independent of the transversal pair (M, N) representing the homology classes $([M], [N])$. Assuming that we can extend the operation $\cap$ to arbitrary homology classes in X, we would obtain an operator between the homology groups $H_k(X, \mathbb{Z})$ with respect to which the degree is not additive. Moreover, the appearance of $\dim X$ in the formula indicates that such an operator can not be defined for arbitrary topological spaces. One of the motivations for studying cohomology groups below is to define homology-type groups that admit a ring structure.

Homology groups arise from chain complexes. In a chain complex, the boundary map is a degree decreasing operator. When studying algebraic structures associated to topological problems, we sometimes obtain a sequence of degree increasing operators between groups indexed by integers in the following sense.

Definition 9.1. A cochain complex is a sequence of degree increasing coboundary maps

$$\cdots C^{n-1} \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \xrightarrow{\partial^{n+1}} \cdots$$

between commutative groups such that $\partial^{n+1}\partial^n = 0$ for all $n \in \mathbb{Z}$.

Usually, we have $C^n \equiv 0$ for $n < 0$ and the sequence starts with C^0. The following shows that there is a natural way to transform any chain complex to a co-chain complex and vice-versa.

Definition 9.2. Given a chain complex

$$\cdots C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots ,$$

and an abelian group G, the cochain complex $(C^\bullet, \partial^*)$ dual to $(C_\bullet, \partial)$ consists abelian groups

$$C^n = C_n^* := \mathrm{Hom}(C_n, G)$$

and coboundary maps

$$\partial^n := \partial_n^* \colon \mathrm{Hom}(C_n, G) \longrightarrow \mathrm{Hom}_{\mathcal{R}}(C_{n+1}, G), \qquad \partial^n(\varrho) := \varrho \circ \partial_{n+1}. \tag{9.1}$$

The converse is defined similarly. If C_n and G are $\mathcal{R}$-modules, we can also consider the dual chain complex of $\mathcal{R}$-modules with $C_n^* := \mathrm{Hom}_{\mathcal{R}}(C^n, G)$. The case where $\mathcal{R} = \mathbb{Z}$ corresponds to arbitrary abelian groups.

Remark 9.3. Note that, by definition

$$\partial_{n+1}^* \partial_n^*(\varrho) = \partial_{n+1}^*\big(\partial_n^*(\varrho)\big) = \partial_{n+1}^*\big(\varrho \circ \partial_{n+1}\big) = \varrho \circ \partial_{n+1} \circ \partial_{n+2} = 0;$$

i.e. $\partial_{n+1}^* \partial_n^* = 0$.

Remark 9.4. Note that applying the duality twice we don't necessarily get the original chain/cochain complex. For instance, if $C = \mathbb{Z}_2$, then

$$C^* = \operatorname{Hom}(\mathbb{Z}_2, \mathbb{Z}) = 0 \quad \text{and} \quad C^{**} = \operatorname{Hom}(C^*, \mathbb{Z}) = 0.$$

Therefore, $C^{**} \neq C$. However, if $\mathcal{R}$ is a field and C_n are $\mathcal{R}$-vector spaces, then $C_n^* := \operatorname{Hom}_{\mathcal{R}}(C_n, \mathcal{R})$ is the dual vector space and C_n^{**} is naturally isomorphic to C_n. Therefore, there will be no loss of information in this process.

The cohomology groups of a cochain complex $(C^\bullet, \partial^\bullet)$ are the quotient spaces ($\mathcal{R}$-modules)

$$H^k(C^\bullet, \partial^\bullet) = \frac{\operatorname{Ker}\big(\partial_k^\bullet \colon C^k \longrightarrow C^{k+1}\big)}{\operatorname{Im}\big(\partial_{k-1}^\bullet \colon C^{k-1} \longrightarrow C^k\big)} \qquad \forall\, k \in \mathbb{Z}.$$

Definition 9.5. Simplicial, singular, and cellular cohomology groups of a simplicial complex/topological space/CW complex X with coefficients in a commutative group G are the cohomology groups of the cochain complexes dual to the integral simplicial, singular, and cellular chain complexes constructed using $\operatorname{Hom}(-, G)$.

It is natural to wonder if there is any relation between the homology groups of a chain complex and the cohomology groups of it dual cochain complex. The following lemma is a sign of hope.

Lemma 9.6. *For every chain complex (C, ∂) and every abelian group G, with $C^n := \operatorname{Hom}(C_n, G)$ for all $n \in \mathbb{Z}$, there are natural homomorphisms*

$$D_n \colon H^n(C^\bullet, \partial^\bullet) \longrightarrow \operatorname{Hom}\big(H_n(C_\bullet, \partial_\bullet), G\big).$$

Furthermore, if C_n are free $\mathbb{Z}$-modules then D_n is surjective and has a (non-canonical) right-inverse.

In the light of the lemma above, one may ask whether D_k is an isomorphism; i.e.

$$H^n(C^\bullet, \partial^\bullet) \stackrel{?}{=} \operatorname{Hom}\big(H_n(C_\bullet, \partial_\bullet), G\big).$$

The following example shows that the answer is not always yes.

Example 9.7. Let $(C_\bullet, \partial_\bullet)$ be the chain complex

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0,$$

and $G = \mathbb{Z}$. Then, the dual cochain complex $(C^\bullet, \partial^\bullet)$ is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0.$$

As a sequence, both are the same. However, they are graded differently. The former ends at C_0 and the latter begins with C^0. Thus, we have

$$H_0(C_\bullet, \partial_\bullet) = H^3(C^\bullet, \partial^\bullet) = \mathbb{Z}, \quad H_1(C_\bullet, \partial_\bullet) = H^2(C^\bullet, \partial^\bullet) = \mathbb{Z}_2,$$
$$H_2(C_\bullet, \partial_\bullet) = H^1(C^\bullet, \partial^\bullet) = 0, \quad H_3(C_\bullet, \partial_\bullet) = H^0(C^\bullet, \partial^\bullet) = \mathbb{Z}.$$

The following lemma explains the loss of exactness in passing to dual.

Lemma 9.8. *The functor* $\mathrm{Hom}(-,G)$ *is only left-exact in the sense that if*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0 \tag{9.2}$$

is a short exact sequence then

$$0 \longrightarrow \mathrm{Hom}(C,G) \xrightarrow{j^*} \mathrm{Hom}(B,G) \xrightarrow{i^*} \mathrm{Hom}(A,G) \tag{9.3}$$

is exact but the last map is not necessarily surjective (nevertheless, it is still a cochain complex).

Example 9.9. The $\mathbb{Z}$-dual of the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}_n \longrightarrow 0,$$

is

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z}.$$

The latter has a non-trivial cohomology group $H^2 = \mathbb{Z}_n$.

In the following, first, we prove Lemma 9.6 and then we delve into some details about the Hom functor $\mathrm{Hom}(-,G)$, how the left-exact sequence (9.3) can be completed on the right, and its connection to understanding the Kernel (and CoKernel) of D_n.

Proof of Lemma 9.6. By (9.1), $\varrho \in C_n^* := \mathrm{Hom}(C_n, G)$ is a cochain complex, i.e. $\partial_n^*\varrho = 0$, if and only if $\varrho \circ \partial_{n+1} = 0$. Since ϱ vanishes on the image of ∂_{n+1}, it descends to a homomorphism

$$D_n(\varrho)\colon H_n(C_\bullet, \partial_\bullet) = \frac{\mathrm{Ker}\big(\partial_n\big)}{\mathrm{Im}\big(\partial_{n+1}\big)} \longrightarrow G. \tag{9.4}$$

Furthermore, if $\varrho \in \mathrm{Im}(\partial_{n-1}^*)$, then $\varrho = \varrho' \circ \partial_n$ for some $\varrho' \in \mathrm{Hom}_{\mathcal{R}}(C_{n-1}, G)$. Therefore, $D_n(\varrho)$ vanishes on $\mathrm{Ker}\big(\partial_n\big)/\mathrm{Im}\big(\partial_{n+1}\big)$. We conclude that (9.4) only depends on the cohomology class $[\varrho]$ of ϱ.

Note that every homomorphism

$$\eta\colon H_n(C, \partial) \longrightarrow G$$

can be seen as a homomorphism

$$\eta\colon \mathrm{Ker}\big(\partial_n\big) \subset C_n \longrightarrow G$$

that vanishes on $\mathrm{Im}\big(\partial_{n+1}\big)$. Therefore, by (9.1), for any extension

$$\widetilde{\eta}\colon C_n \longrightarrow G$$

of η we automatically have $\partial_n^*\widetilde{\eta} = 0$. In general, the short exact sequence

$$0 \longrightarrow \mathrm{Ker}\big(\partial_n\big) \longrightarrow C_n \longrightarrow \mathrm{Im}\big(\partial_{n+1}\big) \longrightarrow 0 \tag{9.5}$$

need not be split. However, if C_{n+1} is free, then $\mathrm{Im}\big(\partial_{n+1}\big)$ is also free and the sequence splits. Therefore, every $\eta \in \mathrm{Ker}\big(\partial_n\big)$ has an extension to C_n and we conclude that D_n is surjective. The construction of $\widetilde{\eta}$ also shows that a choice of splitting in (9.5) results in a right-inverse for D_n. $\square$

Exercise 9.10. If $\mathcal{R}$ is a field and C_n are $\mathcal{R}$-vector spaces, then $C_n^* = \mathrm{Hom}_{\mathcal{R}}(C_n, \mathcal{R})$ are the dual vector spaces. In this situation, prove that D_n is an isomorphism for all n.

Example 9.11. Let us discuss the example of simplicial cohomology with coefficients in an abelian group G. Suppose $\mathcal{K}$ is a simplicial complex with the set of n-simplices $\{\Delta_\alpha\}_{\alpha\in\mathcal{I}_n}$. Then

$$C^n = \bigoplus_{\alpha\in\mathcal{I}_n} G \cdot \Delta_\alpha,$$

where we fix an orientation on each simplex, and $\partial^\bullet$ are defined in the following way. We have

$$\partial^n \Delta_\alpha = \sum_{\beta\in\mathcal{I}_{n+1}\colon\, \Delta_\alpha\in\partial^{\mathrm{top}}\Delta_\beta} (-1)^{\varepsilon_{\alpha,\beta}},$$

where $\varepsilon_{\alpha,\beta} = 1$ if and only if the orientation on Δ_α is the same as the boundary orientation coming from the orientation on Δ_β. It is also important to notice that if Δ_α appears as a boundary of Δ_β is multiple ways (for instance when the two ends of an interval are attached at a point), then each way contributes one summand to the sum on the right.

Going back to Lemma 9.8, note that if the sequence (9.2) splits, then (9.3) will be exact at right as well. As we noted in the proof of Lemma 9.6, this is for instance the case when C is free.

Definition 9.12. Suppose A is an abelian group (there are similar versions for $\mathcal{R}$-modules). A free resolution of A is an exact sequence

$$\cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} A \longrightarrow 0$$

such that F_i are free groups.

By making the last arrow vertical and add a sequence of zero groups and trivial maps under F_i with $i > 0$, we get the commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d] & F_1 \arrow[r, "f_1"] \arrow[d] & F_0 \arrow[d, "f_0"] \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & A \end{tikzcd} \tag{9.6}$$

which is a map of chain complexes; one chain complex is

$$(F_\bullet, f_\bullet) := \big(\cdots \longrightarrow F_2 \xrightarrow{f_2} F_1 \xrightarrow{f_1} F_0 \longrightarrow 0\big)$$

whose only non-trivial homology is $H_0(F_\bullet, f_\bullet) = A$ and the other one is the chain complex with a single nontrivial term A in degree 0 and $H_0 = A$. In other words, a free resolution replaces one abelian group with a sequence of free abelian groups whose homology is A. The map f_0 together with trivial maps in higher degrees defines a chain map $f_0\colon (F_\bullet, f_\bullet) \longrightarrow A$.

Note that if A is free, we can take the obvious resolution $F_0 = A$ and $F_i = 0$ for all $i > 0$. Every abelian group (or finitely generated modules over a P.I.D) has a two-term free resolution

$$0 \longrightarrow F_1 \xrightarrow{f_1} F_0 \longrightarrow A \longrightarrow 0$$

where F_0 is the free group over a set of generators. Free resolutions with more terms appear in some examples or other homological contexts. Note that if A is not a discrete group such as $U(1)$, F_0 and F_1 will involve uncountably many generators. Fortunately, here, we mainly concern finitely generated abelian groups which are direct sum of a bunch of $\mathbb{Z}$ and $\mathbb{Z}/m\mathbb{Z}$.

Lemma 9.13. *Let A be an abelian group.*

(1) *Given abelian groups A and A', $\alpha \in \mathrm{Hom}(A, A')$, and free resolutions $f_0\colon (F_\bullet, f_\bullet) \longrightarrow A$ and $f_0'\colon (F_\bullet', f_\bullet') \longrightarrow A'$, there are homomorphisms $\alpha_i \in \mathrm{Hom}(F_i, F_i')$, for all $i \geq 0$, such that the following diagram commutes*

$$\begin{tikzcd} \cdots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & F_0 \arrow[r, "f_0"] \arrow[d, "\alpha_0"] & A \arrow[d, "\alpha"] \\ \cdots \arrow[r] & F_2' \arrow[r, "f_2'"] & F_1' \arrow[r, "f_1'"] & F_0' \arrow[r, "f_0'"] & A' \end{tikzcd}$$

Different choices of chain maps $\alpha_\bullet := (\alpha_i)_{i\in\mathbb{N}}\colon (F_\bullet, f_\bullet) \longrightarrow (F_\bullet', f_\bullet')$ are chain homotopic.

(2) *Every two resolutions of A are homotopy equivalent.*

Proof. Part (2) follows from part (1) in the following way. For $A = A'$, $\alpha = \mathrm{id}$, and two free resolutions $f_0\colon (F_\bullet, f_\bullet) \longrightarrow A$ and $f_0'\colon (F_\bullet', f_\bullet') \longrightarrow A$, we get chain maps

$$\alpha_\bullet\colon (F_\bullet, f_\bullet) \longrightarrow (F_\bullet', f_\bullet') \quad \text{and} \quad \beta_\bullet\colon (F_\bullet', f_\bullet) \longrightarrow (F_\bullet, f_\bullet).$$

Both

$$\beta_\bullet \circ \alpha_\bullet\colon (F_\bullet, f_\bullet) \longrightarrow (F_\bullet, f_\bullet) \quad \text{and} \quad \alpha_\bullet \circ \beta_\bullet\colon (F_\bullet', f_\bullet') \longrightarrow (F_\bullet', f_\bullet')$$

extend the identity map on A. So they are chain homotopic to trivial extensions $\mathrm{id}\colon (F_\bullet, f_\bullet) \longrightarrow (F_\bullet, f_\bullet)$ and $\mathrm{id}\colon (F_\bullet', f_\bullet') \longrightarrow (F_\bullet', f_\bullet')$.

For part (1), lets first build α_0. For each generator x of F_0, since f_0' is surjective, there is $y \in F_0'$ such that $f_(' y) = \alpha(f_0(x))$. Choose one such y and define $\alpha_0(x) = y$. Making similar choices over a set of generators of F_0 defines α_0. The exactness of the rows imply $\mathrm{Im}(f_1) = \mathrm{Ker}(f_0)$ and $\mathrm{Im}(f_1') = \mathrm{Ker}(f_0')$. The commutativity of the right square implies $\mathrm{Im}(\alpha_0) \subset \mathrm{Ker}(f_0') = \mathrm{Im}(f_1')$. Replacing $\alpha\colon A \longrightarrow A'$ with $\alpha_0\colon \mathrm{Im}(f_0)\colon \mathrm{Im}(f_0')$, the existence of α_1 filling the diagram below follows from the argument above,

$$\begin{tikzcd} \cdots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1"] & \mathrm{Im}(f_0) \arrow[d, "\alpha_0"] \\ \cdots \arrow[r] & F_2' \arrow[r, "f_2'"] & F_1' \arrow[r, "f_1'"] & \mathrm{Im}(f_0') . \end{tikzcd}$$

Inductively, we can eliminate one square at a time and build all the α_i from right to left.

To show that different choices of chain maps $(\alpha_i)_{i\in\mathbb{N}}\colon (F_\bullet, f_\bullet) \longrightarrow (F_\bullet', f_\bullet')$ are chain homotopic is equivalent to showing that if $\alpha = 0$, then every such $(\alpha_i)_{i\in\mathbb{N}}$ is homotopic to the trivial chain map $(0)_{i\in\mathbb{N}}$. In other words, if $\alpha = 0$, we find $h_i\colon F_i \longrightarrow F_{i+1}'$ (including $h_{-1}\colon F_{-1} = A \longrightarrow F_0$) such that $f_{i+1}' h_i + f_i h_{i-1} = \alpha_i$ for all $i \geq -1$ (with $\alpha_{-1} = \alpha = 0$). We take $h_{-1} = 0$. We have

$$\mathrm{Im}(\alpha_0) \subset \mathrm{Ker}(f_0') = \mathrm{Im}(f_1').$$

Therefore, for every $x \in F_0$, there is $y \in F_1'$ such that $\alpha_0(x) = f_1'(y)$. Fix a basis for F_0 and define h_0 to be the map $x \longrightarrow y$ for some choice of y as above. By definition, we have

$f_1'h_0 + h_{-1}f_0 = \alpha_0$. Next, consider the $\alpha_1' = \alpha_1 - h_0 f_1 \colon F_1 \longrightarrow F_1'$. The diagram

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "f_2"] \arrow[d, "\alpha_2"] & F_1 \arrow[r, "f_1"] \arrow[d, "\alpha_1'"] & \mathrm{Im}(f_0) \arrow[d, "0"] \\
\cdots \arrow[r] & F_2' \arrow[r, "f_2'"] & F_1' \arrow[r, "f_1'"] & \mathrm{Im}(f_0') .
\end{tikzcd}$$

commutes. Therefore, applying the previous argument we get $h_1 \colon \mathrm{Im}(f_0) \longrightarrow F_1'$ such that $f_1'h_1 = \alpha_1' = \alpha_1 - h_0 f_1$; in other words $\alpha_1 = f_2'h_1 + h_0 f_1$. Inductively, we can eliminate one square at a time and build all the h_i from right to left. $\square$

If A and G are abelian groups and $(F_\bullet, f_\bullet)$ is a free resolution of A, applying $\mathrm{Hom}(-, G)$ to the exact chain complex $(F_\bullet, f_\bullet)$ we obtain the cochain complex

$$0 \longrightarrow \mathrm{Hom}(F_0, G) \longrightarrow \mathrm{Hom}(F_1, G) \longrightarrow \cdots$$

which is not necessarily exact anymore. Let $\mathrm{Ext}^i(A, G)$ be the i-th cohomology group of this sequence.

Lemma 9.14. *The groups* $\mathrm{Ext}^i(A, G)$ *are independent of the choice of the free resolution involved (so the notation is justified). We have* $\mathrm{Ext}^0(A, G) = \mathrm{Hom}(A, G)$ *and* $\mathrm{Ext}^i(A, G) = 0$ *for* $i > 2$.

Proof. The first statement is a consequence of Lemma 9.13.(2). We know every abelian group A has a two-term free resolution

$$0 \longrightarrow F_1 \xrightarrow{f_1} F_0 \xrightarrow{f_0} A \longrightarrow 0,$$

which is a short-exact sequence. By Lemma 9.8, applying $\mathrm{Hom}(-, G)$, we obtain the left-exact sequence

$$0 \longrightarrow \mathrm{Hom}(A, G) \xrightarrow{f_0^*} \mathrm{Hom}(F_0, G) \xrightarrow{f_1^*} \mathrm{Hom}(F_1, G). \tag{9.7}$$

We conclude that $\mathrm{Ext}^0(A, G) = \ker(f_1^*) = \mathrm{Hom}(A, G)$ and $\mathrm{Ext}^i(A, G) = 0$ for $i > 1$, and $\mathrm{Ext}^1(A, G) = \mathrm{Coker}(f_1^*)$. $\square$

By the proof above, $\mathrm{Ext}^1(A, G)$ is the group which completes the left-exact sequence (9.7) into an exact sequence

$$0 \longrightarrow \mathrm{Hom}(A, G) \xrightarrow{f_0^*} \mathrm{Hom}(F_0, G) \xrightarrow{f_1^*} \mathrm{Hom}(F_1, G) \longrightarrow \mathrm{Ext}^1(A, G) \longrightarrow 0.$$

Remark 9.15. The functors $\mathrm{Ext}^i(-, G)$ are called the derived functors of the functor $\mathrm{Hom}(-, G)$. As we mentioned before, these functors exist in much more generality than used here. In other contexts, the higher Ext functors (beyond Ext^1) can be non-trivial. As the following lemma shows, Ext^1 is related to group extensions. Note that if A is free then $\mathrm{Ext}^1(A, G) = 0$.

Exercise 9.16. Find $\mathrm{Ext}^1(\mathbb{Z}_m, \mathbb{Z}_n)$.

Theorem 9.17. *(i) A short exact sequence of abelian groups*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

results in a long-exact sequence

$$0 \to \mathrm{Hom}(C,G) \to \mathrm{Hom}(B,G) \to \mathrm{Hom}(A,G) \to \mathrm{Ext}^1(C,G) \to \mathrm{Ext}^1(B,G) \to \mathrm{Ext}^1(A,G) \to 0.$$

(ii) There is a one-to-one correspondence between $\mathrm{Ext}^1(C,A)$ *and the set of isomorphism classes of extensions (short-exact sequencess)*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

of C *by* A*. The trivial extension* $B = A \oplus C$ *corresponds to* $0 \in \mathrm{Ext}^1(C,A)$*.*

Exercise 9.18. To prove part (i) of Theorem 9.17, show that given two-term free resolutions $0 \to F_1 \to F_0 \to A \to 0$ and $0 \to H_1 \to H_0 \to C \to 0$, we can build a two-term free resolution of B of the form

$$0 \to F_1 \oplus H_1 \to F_0 \oplus H_0 \to B \to 0.$$

Then, use the short exact sequence $0 \to F_\bullet \to (F_\bullet + H_\bullet) \to H_\bullet \to 0$ of chain complexes to prove thee desired result.

Remark 9.19. By part (i) of the theorem above, applying $\mathrm{Hom}(-,A)$ to an extension, we get a long-exact sequence

$$0 \to \mathrm{Hom}(C,A) \xrightarrow{j^*} \mathrm{Hom}(B,A) \xrightarrow{i^*} \mathrm{Hom}(A,A) \xrightarrow{\delta} \mathrm{Ext}^1(C,A) \to \mathrm{Ext}^1(B,A) \to \mathrm{Ext}^1(A,A) \to 0.$$

This long-exact sequence allows us to define a map

$$e\colon \{\text{the set of isomorphism classes of extensions of } C \text{ by } A\} \to \mathrm{Ext}^1(C,A)$$

by sending B to

$$e(B) = \delta(\mathrm{id}_A) \in \mathrm{Ext}^1(C,A).$$

Part (ii) claims that e is an isomorphism; see Theorem 3.4.3 from Weibel's book: "Introduction to homological algebra".

Exercise 9.20. Find an additive group structure on the set of isomorphism classes of extensions of C by A that corresponds to the homological group structure of $\mathrm{Ext}^1(C,A)$ (There is a construction known as Baer sum). What does the multiplication by -1 in $\mathrm{Ext}^1(C,A)$ correspond to?

We are ready to state and prove the result that explains the kernel of D_n (whenever the chain complex consists of free groups).

Theorem 9.21. *(Universal Coefficient Theorem) Let* $(C_\bullet, \partial_\bullet)$ *be a chain complex of free abelian groups,* G *be an abelian group and* $(C^\bullet, \partial^\bullet) = \mathrm{Hom}\big((C_\bullet, \partial_\bullet), G\big)$*. Then there are natural short exact sequences*

$$0 \longrightarrow \mathrm{Ext}^1\big(H_{n-1}(C_\bullet, \partial_\bullet), G\big) \longrightarrow H^n(C^\bullet, \partial^\bullet) \longrightarrow \mathrm{Hom}(H_n(C_\bullet, \partial_\bullet), G) \longrightarrow 0$$

for all n*. These sequences split, but the splitting is not natural.*

Proof. For every n, let $Z_n \subset C_n$ and $B_{n-1} \subset C_{n-1}$ denote the kernel and image of $\partial_n \colon C_n \longrightarrow C_{n-1}$. Since, by assumption, each C_n is free, Z_n and B_n are free for all n. We have a short exact sequence of chain complexes

$$\begin{tikzcd}
\cdots \arrow[r] & Z_{n+1} \arrow[r, "0"] \arrow[d, "i_{n+1}"] & Z_n \arrow[r, "0"] \arrow[d, "i_n"] & Z_{n-1} \arrow[r, "0"] \arrow[d, "i_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "j_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "j_n"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "j_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_n \arrow[r, "0"] & B_{n-1} \arrow[r, "0"] & B_{n-2} \arrow[r, "0"] & \cdots
\end{tikzcd}$$

which we shortly write as $0 \longrightarrow (Z_\bullet, 0) \longrightarrow (C_\bullet, \partial_\bullet) \longrightarrow (B_{\bullet-1}, 0) \longrightarrow 0$. Since all groups in these chain complexes are free, applying the functor $\mathrm{Hom}(-, G)$ yields again a short exact sequence of cochain complexes

$$0 \longrightarrow \big(\mathrm{Hom}(B_{\bullet-1}, G), 0\big) \longrightarrow \big(\mathrm{Hom}(C_\bullet, G), \partial^\bullet\big) \longrightarrow \big(\mathrm{Hom}(Z_\bullet, G), 0\big) \longrightarrow 0.$$

Since the differentials in $\mathrm{Hom}(B_{\bullet-1}, G)$ and $\mathrm{Hom}(Z_\bullet, G)$ are trivial, the n-th cohomology of the former is just $\mathrm{Hom}(B_{n-1}, G)$, and the n-th cohomology of the latter is just $\mathrm{Hom}(Z_n, G)$. Therefore, the long exact sequence in cohomology associated to the short exact sequence $0 \to (Z_\bullet, 0) \to (C_\bullet, \partial_\bullet) \to (B_{\bullet-1}, 0) \to 0$ reads

$$\cdots \to \mathrm{Hom}(Z_{n-1}, G) \stackrel{\delta_{n-1}}{\to} \mathrm{Hom}(B_{n-1}, G) \to H^n(C^\bullet, G) \to \mathrm{Hom}(Z_n, G) \stackrel{\delta_n}{\to} \mathrm{Hom}(B_n, G) \to \cdots$$

The following two/three steps complete the proof of the first statement.

- It can be easily shown than that the connecting homomorphism $\mathrm{Hom}(Z_n, G) \stackrel{\delta_n}{\to} \mathrm{Hom}(B_n, G)$ is simply the dual ι_n^* of the n-th inclusion map $\iota_n \colon B_n \hookrightarrow Z_n$.
- The short exact sequences

  $$0 \longrightarrow B_{n-1} \stackrel{\iota_{n-1}}{\longrightarrow} Z_{n-1} \longrightarrow H_{n-1}(C_\bullet, \partial) \longrightarrow 0$$

  and

  $$0 \longrightarrow B_n \stackrel{\iota_n}{\longrightarrow} Z_n \longrightarrow H_n(C_\bullet, \partial) \longrightarrow 0$$

  are free resolutions of $H_{n-1}(C_\bullet, \partial_\bullet)$ and $H_n(C_\bullet, \partial_\bullet)$, respectively. Therefore,

  $$\mathrm{Ext}^1\big(H_{n-1}(C_\bullet, \partial_\bullet), G\big) = \mathrm{Coker}(\iota_{n-1}^*).$$

  and

  $$\mathrm{Hom}(H_n(C_\bullet, \partial_\bullet), G) = \mathrm{Ker}(\iota_n^*).$$

  The last statement was proved in Lemma 9.6.

□

Remark 9.22. The homological version of the Universal Coefficient Theorem is the short exact sequence

$$0 \longrightarrow H_n(C, \partial) \otimes G \longrightarrow H_n(C, G) \longrightarrow \mathrm{Tor}^1(H_{n-1}, G) \longrightarrow 0.$$

Here, (C, ∂) is a chain complex of abelian groups ($\mathbb{Z}$-modules), $H_n(C, G)$ is obtained by tensoring the chain complex with G (over $\mathbb{Z}$) and then taking homology versus $H_n(C, \partial) \otimes G$ where the homology groups are tensored with G. The functor Tor^1 is the derived functor of the functor $A \longrightarrow A \otimes G$. This functor is covariant but right-exact; i.e. if $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is an exact sequence then the following sequence is exact:

$$A \otimes G \longrightarrow B \otimes G \longrightarrow C \otimes G \longrightarrow 0.$$

Just like the Ext functor, Tor functor measures the non-exactness of the sequence above on the left-hand side and allows us to complete it to

$$0 \longrightarrow \mathrm{Tor}^1(A, G) \longrightarrow \mathrm{Tor}^1(B, G) \longrightarrow \mathrm{Tor}^1(C, G) \longrightarrow A \otimes G \longrightarrow B \otimes G \longrightarrow C \otimes G \longrightarrow 0$$

We can use free resolutions to calculate Tor^1.

Remark 9.23. If A is a finitely generated group, then it is a finite direct sum of copies of $\mathbb{Z}$ and $\mathbb{Z}_m$ (for various values of m). Then it can be easily seen that $\mathrm{Hom}(A, \mathbb{Z}) \cong A_{\text{free}}$ is isomorphic the free part of A and $\mathrm{Ext}^1(A, \mathbb{Z}) \cong A_{\text{tor}}$ is isomorphic to the torsion part. Therefore, in the context of the Universal Coefficient Theorem, with $G = \mathbb{Z}$, if $H_n(C, \partial)$ and $H_{n-1}(C, \partial)$ are finitely generated, then

$$H^n(C^\bullet, \partial^\bullet) \cong H_n(C, \partial)_{\text{free}} \oplus H_{n-1}(C, \partial)_{\text{tor}}. \tag{9.8}$$

This appears in Poincare duality and is useful in understanding the effect of chain maps on homology groups vs cohomology groups.

Example 9.24. We can dualize the chain complex in Exercise 8.2 or use (9.8) to show that

$$H^i(\mathbb{RP}^m, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ \mathbb{Z} & \text{if } i = m \text{ and } m \text{ is odd}, \\ 0 & \text{if } 0 < i < m,\ i = \text{odd} \\ \mathbb{Z}_2 & \text{if } 0 < i \leq m,\ i = \text{even}. \end{cases}$$

If $A \subset X$ is a subspace, relative cohomology groups $H^n(X, A, G)$ with coefficients in an abelian G are the cohomology groups of the singular cochain complex

$$(C^\bullet(X, A, G), \partial^\bullet) := \mathrm{Hom}\big((C_\bullet(X, A, \mathbb{Z}), \partial_\bullet), G\big).$$

Here $(C_\bullet(X, A, \mathbb{Z}), \partial_\bullet)$ is the cokernel singular chain complex

$$0 \longrightarrow (C_\bullet(A, \mathbb{Z}), \partial_\bullet) \longrightarrow (C_\bullet(X, \mathbb{Z}), \partial_\bullet) \longrightarrow (C_\bullet(X, A, \mathbb{Z}), \partial_\bullet) \longrightarrow 0.$$

Since these are chain complexes of free abelian groups, applying $\mathrm{Hom}\big(-, G\big)$ preserves the exactness and we obtain the dual short exact sequence

$$0 \longrightarrow (C^\bullet(X, A, G), \partial^\bullet) \longrightarrow (C^\bullet(X, G), \partial^\bullet) \longrightarrow (C^\bullet(A, G), \partial^\bullet) \longrightarrow 0.$$

The long-exact sequence of the relative cohomology groups associated this short-exact sequence of cochain complexes reads

$$\cdots \longrightarrow H^n(X, A, G) \longrightarrow H^n(X, G) \longrightarrow H^n(A, G) \longrightarrow H^{n+1}(X, A, G) \longrightarrow \cdots.$$

In Section 4, we learned that every continuous map $f\colon X \longrightarrow Y$ induces a push-forward homomorphism $f_*\colon H_n(X) \longrightarrow H_n(X)$ between the homology groups. Furthermore, we showed that f_* is an isomorphism whenever f is a homotopy equivalence. The map f_* is obtained from the composition of singular chains in X and f. Dualizing this process with $\mathrm{Hom}(-, G)$, we obtain a pull-back map

$$f^*\colon H_n(Y, G) \longrightarrow H_n(X, G)$$

in the reverse direction. Again, f^* is an isomorphism whenever f is a homotopy equivalence. The maps f^* and

$$(f_*)^*\colon \mathrm{Hom}(H_n(Y), G) \longrightarrow \mathrm{Hom}(H_n(X), G)$$

are related by the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1\big(H_{n-1}(X), G\big) \arrow[r] & H^n(X, G) \arrow[r] & \mathrm{Hom}(H_n(X), G) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}^1\big(H_{n-1}(Y), G\big) \arrow[r] \arrow[u, "(f_*)^*"] & H^n(Y, G) \arrow[r] \arrow[u, "f^*"] & \mathrm{Hom}(H_n(Y), G) \arrow[r] \arrow[u, "(f_*)^*"] & 0\,.
\end{tikzcd}$$

where the rows are Universal Coefficient Theorem on X and Y.

In the relative case, a continuous map of pairs $f\colon (X, A) \longrightarrow (Y, B)$ induces $f^*\colon H_n(Y, B, G) \longrightarrow H_n(X, A, G)$ by the same reasoning. Universal Coefficient Theorem applies also to relative cohomology (since the relative chain groups $C_n(X, A)$ are free) and we get a split short exact sequence

$$0 \longrightarrow \mathrm{Ext}^1\big(H_{n-1}(X, A), G\big) \longrightarrow H^n(X, A, G) \longrightarrow \mathrm{Hom}(H_n(X, A), G) \longrightarrow 0$$

for all n.

Finally, the cohomological version of Mayer-Vietoris long-exact sequence for a decomposition $X = A \cup B$ (such that the inertoris of A and B cover X) reads

$$\cdots \longrightarrow H^n(X, G) \longrightarrow H^n(A, G) \oplus H^n(B, G) \longrightarrow H_n(A \cap B, G) \longrightarrow H^{n+1}(X, G) \longrightarrow \cdots\,.$$

10 Ring structure of the singular cohomology

Given topological spaces X and Y and ring $\mathcal{R}$ (which can be considered as an abelian group with respect to the addition map), we define a degree-compatible product map

$$\times\colon H^n(X, \mathcal{R}) \otimes_{\mathcal{R}} H^m(Y, \mathcal{R}) \longrightarrow H^{n+m}(X \times Y, \mathcal{R}). \tag{10.1}$$

In fact, we define $\times$ as map from the products of singular cochain complexes of X and Y to the cochain complex of $X \times Y$. For this to work out, we need to adopt Definition 3.2 that avoids certain symmetries. When $X = Y$, the diagonal embedding

$$\Delta\colon X \longrightarrow X \times X$$

induces a pull-back map

$$\Delta^*\colon H^n(X \times X, \mathcal{R}) \longrightarrow H^n(X, \mathcal{R}).$$

Combining Δ and $\times$, we obtain the so-called cup product map

$$\smile := \Delta^* \circ \times \colon H^n(X,\mathcal{R}) \times H^m(Y,\mathcal{R}) \longrightarrow H^{n+m}(X,\mathcal{R}). \tag{10.2}$$

Since both Δ^* and $\times$ are induced by map of cochain complexes, $\smile$ is also induced by a bi-linear map from the singular cochain complex of X to itself. We will give an explicit description of this map. Finally, we will study the ring structure of the singular cohomology group with respect to the binary operations $(+,\smile)$.

Let

$$\varphi_X \in C^n(X,\mathcal{R}) = \mathrm{Hom}(C_n(X,\mathbb{Z}),\mathcal{R}) \quad \text{and} \quad \varphi_Y \in C^m(Y,\mathcal{R}) = \mathrm{Hom}(C_m(Y,\mathbb{Z}),\mathcal{R})$$

where $C_\bullet(X,\mathbb{Z})$ and $C_\bullet(Y,\mathbb{Z})$ are the singular chain complexes of X and Y with integral coefficients. Following Definition 3.2, since $C_{n+m}(X\times Y,\mathbb{Z})$ is freely generated by (oriented) singular $n+m$ simplices

$$\sigma\colon \Delta = \Delta_{[v_0,\dots,v_{n+m}]} \longrightarrow X \times Y,$$

in order to define the product

$$\varphi \times \psi \in C^{n+m}(X\times Y,\mathcal{R}) = \mathrm{Hom}(C_{n+m}(X\times Y,\mathbb{Z}),\mathcal{R}),$$

it is enough to specify its value

$$(\varphi_X \times \varphi_Y)(\sigma) \in \mathcal{R}$$

on each σ. Decompose $\sigma = (\sigma_X, \sigma_Y)$ into its X and Y components that correspond to canonical projection maps $X\times Y \longrightarrow X$ and $X \times Y \longrightarrow Y$. Let

$$(\varphi_X \times \varphi_Y)(\sigma) = \varphi_X(\sigma_X|_{\Delta_{[v_0,\dots,v_n]}}) \cdot \varphi_Y(\sigma_Y|_{\Delta_{[v_n,\dots,v_{n+m}]}})$$

Note that the righthand side is where the product (ring) structure on $\mathcal{R}$ is used. It is clear from the definition that $\times$ is an $\mathcal{R}$-bilinear map of $\mathcal{R}$-modules

$$C^n(X,\mathcal{R}) \otimes_{\mathcal{R}} C^m(Y,\mathcal{R}) \longrightarrow C^{n+m}(X\times Y,\mathcal{R}).$$

Lemma 10.1. *For the definition above, we have*

$$\partial^{n+m}(\varphi_X \times \varphi_Y) = \partial^n(\varphi_X) \times \varphi_Y + (-1)^n \varphi_X \times \partial^m(\varphi_Y).$$

In particular, the product of two cocycles is again a cocycle, and the product of a coboundary with a cocycle (in either order) is a coboundary. Thus, the product operation induces a bilinear operation (10.1) on cohomology.

Proof. By definition,

$$\partial^{n+m}(\varphi_X \times \varphi_Y) = (\varphi_X \times \varphi_Y) \circ \partial_{n+m+1}.$$

Therefore, for any singular $(n+m+1)$-simplex

$$\sigma\colon \Delta_{[v_0,\dots,v_{n+m+1}]} \longrightarrow X \times Y$$

we have

$$\begin{aligned}
\big(\partial^{n+m}(\varphi_X\times\varphi_Y)\big)(\sigma) &= (\varphi_X\times\varphi_Y)(\partial_{n+m+1}\sigma)\\
&= (\varphi_X\times\varphi_Y)\Big(\sum_{i=0}^{n+m+1}(-1)^i\sigma|_{\Delta_{[v_0,\dots\widehat{v_i}\dots,v_{n+m+1}]}}\Big)\\
&= \Big(\sum_{i=0}^{n}(-1)^i\varphi_X(\sigma_X|_{\Delta_{[v_0,\dots\widehat{v_i}\dots v_{n+1}]}})\cdot\varphi_Y(\sigma_Y|_{\Delta_{[v_{n+1},\dots,v_{n+m+1}]}})\Big)\\
&+\Big(\sum_{i=n+1}^{n+m+1}(-1)^i\varphi_X(\sigma_X|_{\Delta_{[v_0,\dots\widehat{v_i}\dots v_n]}})\cdot\varphi_Y(\sigma_Y|_{\Delta_{[v_n,\dots\widehat{v_i}\dots,v_{n+m}]}})\Big)\\
&= \Big(\sum_{i=0}^{n+1}(-1)^i\varphi_X(\sigma_X|_{\Delta_{[v_0,\dots\widehat{v_i}\dots v_{n+1}]}})\cdot\varphi_Y(\sigma_Y|_{\Delta_{[v_{n+1},\dots,v_{n+m+1}]}})\Big)\\
&+\Big(\sum_{i=n}^{n+m+1}(-1)^i\varphi_X(\sigma_X|_{\Delta_{[v_0,\dots\widehat{v_i}\dots v_n]}})\cdot\varphi_Y(\sigma_Y|_{\Delta_{[v_n,\dots\widehat{v_i}\dots,v_{n+m}]}})\Big)\\
&= \varphi_X(\partial_{n+1}\sigma_X|_{\Delta_{[v_0,\dots,v_{n+1}]}})\cdot\varphi_Y(\sigma_Y|_{\Delta_{[v_{n+1},\dots,v_{n+m+1}]}})\\
&+(-1)^n\varphi_X(\sigma_X|_{\Delta_{[v_0,\dots,v_n]}})\cdot\varphi_Y(\partial_{m+1}\sigma_Y|_{\Delta_{[v_{n+1},\dots,v_{n+m+1}]}})\\
&= \big(\partial^n(\varphi_X)\times\varphi_Y\big)(\sigma)+(-1)^n\big(\varphi_X\times\partial^m(\varphi_Y)\big)(\sigma).
\end{aligned}$$

□

Theorem 10.2. *If $H^k(Y,\mathcal{R})$ is a finitely generated $\mathcal{R}$-module for all $k\ge 0$, then the product map (10.1) is a ring isomorphism. Here, the product structure on $H^*(X,\mathcal{R})\otimes_{\mathcal{R}}H^*(Y,\mathcal{R})$ is*

$$(\Phi_1\otimes\Psi_1)\cdot(\Phi_2\otimes\Psi_2)=(-1)^{\deg(\Phi_2)\deg(\Psi_1)}(\Phi_1\smile\Phi_2)\otimes(\Psi_1\smile\Psi_2)$$

The proof of this theorem involves (1) considering CW approximations to arbitrary spaces to reduce the problem to CW complexes (2) reduction to finite dimensional CW complexes, (3) induction on the dimension of CW complexes, (4) using Relative long-exact sequence to reduce to relative cohomology groups $H^k(X^{(n)},X^{(n-1)})$ and $H^\ell(Y^{(m)},X^{(m-1)})$, and (5) Explicit calculations on $X^{(n)}/X^{(n-1)}$ which is a wedge sum of spheres. We skip the full proof here and refer to Hatcher. When $\mathcal{R}$ is a field, we provide a short proof with a stronger finiteness condition on Y at the end of this section.

Given a cohomology class $\eta\in H^n(X\times Y,\mathcal{R})$, the decomposition

$$\eta=\sum_{i=0}^{n}\alpha_i\otimes\beta_{n-i},\qquad \alpha_i\in H^i(X,\mathcal{R}),\ \beta_i\in H^i(Y,\mathcal{R})\quad\forall\ i=0,\dots,n,$$

is called the **Künneth** decomposition of η.

Remark 10.3. The cup/cross product between relative cohomology classes is defined similarly.

The following is the relative version of Theorem 10.2.

Theorem 10.4. *If $H^k(Y,B,\mathcal{R})$ is a finitely generated $\mathcal{R}$-module for all $k\ge 0$, then the cross product map*

$$\times\colon H^n(X,A,\mathcal{R})\otimes_{\mathcal{R}}H^m(Y,B,\mathcal{R})\longrightarrow H^{n+m}(X\times Y,A\times Y\cup X\times B,\mathcal{R}).$$

is a ring isomorphism.

Next, let's see a direct definition of the cup product obtained from $\times$ as in (10.2).

It is easy to see that the cup product (10.2) is induced by the cochain-level bilinear map

$$C^n(X,\mathcal{R}) \otimes_{\mathcal{R}} C^m(X,\mathcal{R}) \longrightarrow C^{n+m}(X,\mathcal{R}),$$

that sends (φ,ψ) to $\varphi \smile \psi$ defined by

$$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|_{\Delta_{[v_0,\ldots,v_n]}}) \cdot \psi(\sigma|_{\Delta_{[v_n,\ldots,v_{n+m}]}}). \tag{10.3}$$

Lemma 10.5. *If $\mathcal{R}$ is commutative, then $\Phi \smile \Psi = (-1)^{mn}\Psi \smile \Phi$ for all $\Phi \in H^n(X,\mathcal{R})$ and $\Psi \in H^m(X,\mathcal{R})$.*

Proof. For each n, let

$$c_n \colon C_n(X) \longrightarrow C_n(X)$$

denote the chain map

$$\sigma \longrightarrow (-1)^{\binom{n}{2}}\sigma \circ h_n$$

where h_n is the linear automorphism in (3.1) corresponding to the permutation

$$\tau_n \colon (0,\ldots,n) \longrightarrow (n,\ldots,0).$$

Recall from Remark 3.3 that, since $\mathrm{sign}(\tau_n) = (-1)^{\binom{n}{2}}$, δ_n is chain homotopic to identity and thus induces the identity push-forward map on homology and identity pull-back map on cohomology. For n- and m-cochains φ and ψ representing Φ and Ψ, it is easy to see from (10.3) that

$$c_n^*\varphi \smile c_m^*\psi = (-1)^{nm}c_{n+m}^*(\psi \smile \varphi).$$

Therefore,

$$\Phi \smile \Psi = (-1)^{nm}(\Psi \smile \Phi).$$

$\square$

Before going any further, in two examples, we show how the ring structure can be identified in practice.

Example 10.6. Lets consider a closed oriented Riemann surface Σ of genus g. We have

$$H_0(\Sigma,\mathbb{Z}) \cong \mathbb{Z}, \quad H_2(\Sigma,\mathbb{Z}) \cong \mathbb{Z}, \quad H_1(\Sigma,\mathbb{Z}) \cong \mathbb{Z}^{2g}.$$

Since the homology groups are all free, by Universal Coefficient Theorem, we have

$$H^i(\Sigma,\mathbb{Z}) = \mathrm{Hom}(H_i(\Sigma,\mathbb{Z}),\mathbb{Z}) \qquad \forall\ i = 0,1,2.$$

The surface Σ can be seen as a $2g$-gon with the edges oriented and labeled in counter-clockwise order by $a_1, b_1, a_1, b_1, \cdots, a_g, b_g, a_g, b_g$ as in Figure 3. The orientation of Σ corresponds to counter-clock wise orientation on the polygon. The edges labeled the same are orientation-preservingly identified to make Σ. We further decompose the only 2-cell in the picture into $2g$ triangles to get a Δ-complex structure on Σ. The homology classes of the curves $a_1,\ldots,b_{2g}$ make a $\mathbb{Z}$-basis for $H_1(\Sigma,\mathbb{Z})$. Let $\alpha_1,\beta_1,\ldots,\alpha_{2g},\beta_{2g}$ denote the dual basis for $H^1(\Sigma,\mathbb{Z})$. We show that

$$\alpha_i \smile \alpha_j = \beta_i \smile \beta_j = 0 \quad \text{and} \quad \alpha_i \smile \beta_j = \delta_{ij}\Omega \qquad \forall 1 \le i,j \le g,$$

where Ω is the generator of $H_2(\Sigma,\mathbb{Z})$ dual to the generator $[\Sigma] \in H_2(\Sigma,\mathbb{Z})$.

Details: To be typed

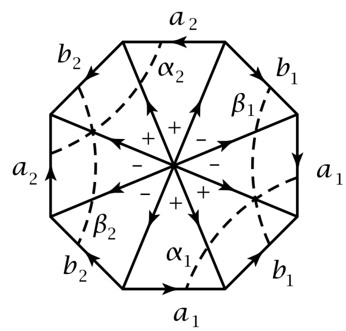

Figure 3: An opened-up genus 2 surface.

Example 10.7. We show that the $\mathbb{Z}_2$-cohomology ring of $\mathbb{RP}^n$ is $\mathbb{Z}[h]/(h^{n+1})$ where h is the generator of $H^1(\mathbb{RP}^n, \mathbb{Z}_2)$. Letting n to go to infinity, we deduce $H^*(\mathbb{RP}^n, \mathbb{Z}_2) = \mathbb{Z}[h]$.

First, it is easy to show that $H^i(\mathbb{RP}^n, \mathbb{Z}_2) \cong \mathbb{Z}_2$. Let h_i denote the generator of this $\mathbb{Z}_2$-vector space. Consider the diagram

$$\begin{tikzcd}
H^i(\mathbb{RP}^n, \mathbb{Z}_2) \times H^{n-i}(\mathbb{RP}^n, \mathbb{Z}_2) \arrow[r, "\smile"] & H^n(\mathbb{RP}^n, \mathbb{Z}_2) \\
H^i(\mathbb{RP}^n, \mathbb{RP}^n - \mathbb{RP}^{n-i}, \mathbb{Z}_2) \times H^{n-i}(\mathbb{RP}^n, \mathbb{RP}^n - \mathbb{RP}^i, \mathbb{Z}_2) \arrow[r, "\smile"] \arrow[u] \arrow[d] & H^n(\mathbb{RP}^n, \mathbb{RP}^n - \{p_i\}, \mathbb{Z}_2) \arrow[u] \arrow[d] \\
H^i(\mathbb{R}^n, \mathbb{RP}^n - \mathbb{R}^{n-i}, \mathbb{Z}_2) \times H^{n-i}(\mathbb{R}^n, \mathbb{R}^n - \mathbb{R}^i, \mathbb{Z}_2) \arrow[r, "\smile"] & H^n(\mathbb{R}^n, \mathbb{R}^n - \{p_i\}, \mathbb{Z}_2)
\end{tikzcd}$$

which commutes by naturality of cup product. Here, $\mathbb{RP}^i$ is the subspace $\{[x_0, \ldots, x_i, 0, \ldots, 0]\} \subset \mathbb{RP}^n$, $\mathbb{RP}^i$ is the complementary subspace $\{[0, \ldots, 0, x_i, \ldots, x_n]\} \subset \mathbb{RP}^n$, $p_i = [0, \ldots, 0, 1, 0 \cdots, 0] = \mathbb{RP}^{n-i} \cap \mathbb{RP}^i$, and the vertical maps are pullbacks. We will show that the four vertical maps are isomorphisms and that the lower cup product map takes generator cross generator to generator. Commutativity of the diagram will then imply that the upper cup product map also takes generator cross generator to generator.

Details: To be typed

We conclude that $h_i \smile h_{n-i} = h_n$. The inclusion $\mathbb{RP}^m \subset \mathbb{RP}^n$, for $m \leq n$, gives a pullback isomorphism $H^i(\mathbb{RP}^n) \longrightarrow H^i(\mathbb{RP}^m)$ for all $i \leq m$. It follows that $h_i \smile h_j = h_{i+j}$ for all $0 \leq i, j \leq n$. Therefore, $h_i = h^i$.

Definition 10.8. We say Y has finite type if Y is a finite union of open sets $Y = U_1 \cup \cdots \cup U_N$ such that each possible intersection of the U_i is either empty or is contractible to a point.

Theorem 10.9 (Weaker version of Theorem 10.2)**.** *If $\mathcal{R}$ is a field and Y has finite type then the product map (10.1) is a ring isomorphism.*

Proof. We prove this theorem using Mayer-Vietoris and induction on N in Definition 10.8. The statement is correct for $N = 1$ because $X \times U_1$ is contractible to X and $H^*(U_1, \mathcal{R}) = H^0(X, \mathcal{R}) = \mathcal{R}$.

Suppose the statement is true for all X and Y, where Y is made of at most $N-1$ such open sets. Suppose $Y = U_1 \cup \cdots \cup U_N$ such that each possible intersection of the U_i is either empty or is contractible to a point. Let

$$Y' = U_1 \cup \cdots \cup U_{N-1}.$$

Note that

$$Y'' = Y' \cap U_{N+1} = (U_1 \cap U_N) \cup \cdots \cup (U_{N-1} \cap U_N)$$

fits into the induction assumption. Corresponding to the decomposition $Y = Y' \cup U_N$, we get the long exact sequence

$$\cdots H^j(Y) \longrightarrow H^j(Y') \oplus H^j(U_N) \longrightarrow H^j(Y'') \longrightarrow \cdot.$$

Since $\mathcal{R}$ is a field, tensoring with $H^i(X)$ preserves exactness and yields a long-exact sequence

$$\cdots (H^i(X) \otimes H^j(Y)) \longrightarrow (H^i(X) \otimes H^j(Y')) \oplus (H^i(X) \otimes H^j(U_N)) \longrightarrow (H^i(X) \otimes H^j(Y'')) \longrightarrow \cdots.$$

Summing over several shifted versions of this sequence we get the long-exact sequence

$$\cdots \bigoplus_{i+j=k} (H^i(X) \otimes H^j(Y)) \longrightarrow \bigoplus_{i+j=k} (H^i(X) \otimes H^j(Y')) \oplus (H^i(X) \otimes H^j(U_N)) \longrightarrow \bigoplus_{i+j=k} (H^i(X) \otimes H^j(Y'')) \longrightarrow \cdots.$$

In the diagram,

$$\begin{tikzcd}
\cdots \bigoplus_{i+j=k}(H^i(X) \otimes H^j(Y)) \arrow[r] \arrow[d, "\times"] & \bigoplus_{i+j=k}(H^i(X) \otimes H^j(Y')) \oplus (H^i(X) \otimes H^j(U_N)) \arrow[r] \arrow[d, "\times"] & \bigoplus_{i+j=k}(H^i(X) \otimes H^j(Y'')) \cdots \arrow[d, "\times"] \\
\cdots H^k(X \times Y) \arrow[r] & H^k(X \times Y') \oplus H^k(X \times U_N) \arrow[r] & \bigoplus_{i+j=k} H^k(X \times Y'') \cdots
\end{tikzcd}$$

for each k, the first and second columns are isomorphisms. By five Lemma, the right column is an isomorphism as well.

□

11 Poincare duality

In Example 10.6, we showed that if X is a Riemann surface of genus 2 (the same argument works in any genus), then every cohomology class $\alpha \in H^1(X, \mathbb{Z})$ corresponds to a homology class $\gamma \in H_1(\Sigma, \mathbb{Z})$ in the sense that

$$\alpha(a) = \gamma \cdot a \in \mathbb{Z} \qquad \forall\ a \in H_1(\Sigma, \mathbb{Z}).$$

On the righthand side, a and γ are represented by transversely intersecting oriented loops in Σ and $\gamma \cdot a$ is the algebraic sum of their intersection points (i.e., # positive intersection - # negative intersections). The definition of intersection number above makes use of X being a closed oriented manifold. Our proof of the isomorphism

$$H^i(X, \mathbb{Z}) \cong H_{2-i}(X, \mathbb{Z})$$

made use of the dual CW complex associated to a simplicial complex structure on X. This proof extends to many closed oriented n-dimensional manifolds (including all smooth oriented): given a nice simplicial complex structure on X, there is a dual CW complex structure on X that can be used to show $H^i(X,\mathbb{Z}) = H_{n-i}(X,\mathbb{Z})$. However, to construct these dual CW structures requires a certain amount of manifold theory. To avoid this, and to get a theorem that applies to all C^0-manifolds, we will take a completely different approach that involves tools/ideas from relative homology theory, covering spaces, and the construction of cup product. The main result of this section is the following.

Theorem 11.1. *Suppose $\mathcal{R}$ is a commutative ring. For every topological space X and $k \geq \ell$, there is a naturally defined cap product map $\frown\colon H_k(X,\mathcal{R}) \otimes_{\mathcal{R}} H^\ell(X,\mathcal{R}) \longrightarrow H_{k-\ell}(X,\mathcal{R})$. If X is a closed $\mathcal{R}$-orientable n-manifold with fundamental class $[X] \in H_n(X,\mathcal{R})$, then*

$$\mathrm{PD}\colon H^\ell(X,\mathcal{R}) \longrightarrow H_{n-\ell}(X,\mathcal{R}), \qquad \alpha \longrightarrow [X] \frown \alpha \in H_{n-\ell}(X,\mathcal{R}),$$

is an isomorphism for all ℓ.

Remark 11.2. Before we delve into the details of this theorem, we provide some comments to help digest the statement of the theorem better.

- Here, "naturally defined" means that cap product satisfies a naturally property with respect to continuous maps $f\colon X \longrightarrow Y$. Given a continuous map $f\colon X \longrightarrow Y$, the relevant induced maps on homology and cohomology fit into the following diagram

$$\begin{tikzcd} H_k(X,\mathcal{R}) \otimes_{\mathcal{R}} H^\ell(X,\mathcal{R}) \arrow[r, "\frown"] \arrow[d, "f_*"] & H_{k-\ell}(X,\mathcal{R}) \arrow[d, "f_*"] \\ H_k(Y,\mathcal{R}) \otimes_{\mathcal{R}} H^\ell(Y,\mathcal{R}) \arrow[u, "f^*"'] \arrow[r, "\frown"] & H_{k-\ell}(Y,\mathcal{R}) \end{tikzcd},$$

  meaning that

$$f_*(A) \frown \alpha = f_*(A \frown f^*(\alpha)) \qquad \forall\ A \in H_k(X,\mathcal{R}),\ \alpha \in H^\ell(Y,\mathcal{R}).$$

  Additionally, we have

$$A \frown (\alpha \smile \beta) = (A \frown \alpha) \frown \beta \qquad \forall\ A \in H_k(X,\mathcal{R}),\ \alpha \in H^\ell(X,\mathcal{R}),\ \beta \in H^{\ell'}(X,\mathcal{R}).$$

  Note that f^* commutes with cup product.

- The notion of $\mathcal{R}$-orientability is a homological incarnation of the usual notion of orientability of smooth manifolds. When $\mathcal{R} = \mathbb{Z}$ and X is smooth, X is $\mathbb{Z}$-oriented iff the top exterior power $\Lambda^{\mathrm{top}}TX$ of the tangent bundle TX is a trivial real line bundle. For continuous manifolds, the tangent bundle is not defined. Nevertheless, the orientability can be captured through comparison of local homology groups. For $\mathcal{R} = \mathbb{Z}_2$, every manifold is $\mathbb{Z}_2$-orientable.

- In the definition of the Poincare duality map PD, $[X]$ is the homology class arising from the obvious inclusion $\iota\colon X \longrightarrow X$.

The cap product in Theorem 11.1 has an explicit equation and is defined as a product map between and chain and cochain complexes of X; for $k \geq \ell$, it is the map

$$\frown\colon C_k(X,\mathcal{R}) \otimes_{\mathcal{R}} C^\ell(X,\mathcal{R}) \longrightarrow C_{k-\ell}(X,\mathcal{R})$$

that sends a pair (σ,φ) of a singular k-simplex $\sigma\colon \Delta_{[v_0,\dots,v_k]}$ and a cochain φ to

$$\sigma \frown \varphi = \varphi(\sigma|_{\Delta_{[v_0,\cdots,v_\ell]}})\sigma|_{\Delta_{[v_\ell,\dots,v_k]}}.$$

Lemma 11.3. *The cap product $\frown$ satisfies*

$$\partial_{k-\ell}(\sigma \frown \varphi) = (-1)^{\ell}(\partial_k \sigma \frown \varphi - \sigma \frown \partial^{\ell}\varphi).$$

In particular, the cap product of a cycle and a cocycle is again a cycle, the product of a boundary with a cocycle is a boundary, and also the product of a cycle with a coboundary is a boundary. Thus, $\frown$ induces a bilinear operation between homology and cohomology groups.

Exercise 11.4. Follow the same idea used in the proof of Lemma 10.1 to prove the lemma above.

This finishes the proof of the first statement of Theorem 11.1. Before we get into the details needed for the proof of the second statement, lets rework Example 10.6

Example 11.5. TBA

If X is an n-manifold, $x \in X$ is a point, and U is an open neighborhood of x homeomorphic to a ball B around the origin in $\mathbb{R}^n$, we have

$$H_n(X, X-\{x\}) \cong H_n(X, X-U) \cong H_n(\mathbb{R}^n, \mathbb{R}^n-\{0\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n-B) \cong H_n(\overline{B}, \partial B = S^{n-1}) \cong \mathcal{R}. \tag{11.1}$$

Definition 11.6. A local $\mathcal{R}$-orientation of X at a point x is a choice of a generator μ_x of $H_n(X, X - \{x\})$. In other words, μ_x is a choice of the isomorphism $\mu_x \colon H_n(X, X - \{x\}) \longrightarrow \mathcal{R}$. When $\mathcal{R} = \mathbb{Z}$, we drop the prefix $\mathcal{R}$- and say μ_x is a local orientation of X at x.

Note that different choices of isomorphism $H_n(X, X - \{x\}) \longrightarrow \mathcal{R}$ differ by a unit in $\mathcal{R}$. If $\mathcal{R} = \mathbb{Z}$, then the subgroup of multiplicative units is $\{\pm 1\}$. So there are two possible choices for each μ_x. When $\mathcal{R} = \mathbb{Z}_2$, however, the subgroup of units consists of only one element μ_x is unique!

Notation. To keep the notation short, we will write $H_n(X \,|\, A)$ instead of $H_n(X, X - A)$, and $H_n(X \,|\, A, \mathcal{R})$ instead of $H_n(X, X - A, \mathcal{R})$.
If x and U are as in (11.1) and $y \in U$, then the composition of the isomorphisms

$$\begin{tikzcd} & H_n(X|U) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ H_n(X|x) & & H_n(X|y) \end{tikzcd}$$

gives an isomorphism $h_U \colon H_n(X|x) \longrightarrow H_n(X|y)$. Therefore, any $\mathcal{R}$-orientation at x induces an $\mathcal{R}$-orientation μ_y at every $y \in U$. In this situation, we say μ_x and μ_y are locally consistent.

Definition 11.7. An $\mathcal{R}$-orientation on a topological n-manifold X is a function $x \longrightarrow \mu_x$, assigning to each x a local $\mathcal{R}$-orientation, such that for each $x \in X$ and some sufficiently small neighborhood $U \ni x$, μ_y and μ_x are consistent for all $y \in U$.

Remark 11.8. The definition above has a formulation in terms of sheaf theory. The assignment

$$U \longrightarrow \text{set of isomorphisms } H_n(X|U) \cong \mathcal{R}$$

is a pre-sheaf. The sheafification of this presheaf is the locally constant sheaf of local orientations. An $\mathcal{R}$-orientation on X is a global section of this sheaf.

Given a topology on the set of units $U(\mathcal{R})$ of $\mathcal{R}$, the set

$$\widetilde{X}_{\mathcal{R}} = \{(x, \mu_x) \colon x \in X,\ \mu_x \text{ is a local } \mathcal{R} - \text{orientation at } x\}$$

has a natural topology such that the projection map

$$\widetilde{X}_{\mathcal{R}} \longrightarrow X, \quad (x, \mu_x) \longrightarrow x,$$

is a continuous $U(\mathcal{R})$-bundle. For instance, if $\mathcal{R} = \mathbb{Z}$, then $U(\mathcal{R}) = \mathbb{Z}_2$ and

$$\widetilde{X} := \widetilde{X}_{\mathbb{Z}} \longrightarrow X \tag{11.2}$$

is the orientable double cover of X. In particular, X is orientable iff $\widetilde{X} = X \sqcup X$ iff $\pi_1(X)$ has no subgroup of index 2.

If $\mathcal{R} = \mathbb{Z}_2$, then

$$\widetilde{X}_{\mathbb{Z}_2} = X.$$

These are the two case that we care mostly about.

Lemma 11.9. *Suppose $\mathcal{R}$ is a commutative ring with an identity element. If X is orientable (i.e., $\mathbb{Z}$-orientable), then it is $\mathcal{R}$-orientable for all $\mathcal{R}$. If X is not orientable, then it is $\mathcal{R}$-orientable if and only if $\mathcal{R}$ contains a unit of order 2.*

Proof. Since $H_n(X|x, \mathcal{R}) = H_n(X|x, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{R}$, each unit $r \in U(\mathcal{R})$ determines a sub-covering $\widetilde{X}_r$ of $\widetilde{X}_{\mathcal{R}}$ consisting of the points $\pm\mu_x \otimes r$ for μ_x a generator of $H_n(X|x, \mathbb{Z})$. If r has order 2, then $r = -r$ and $\widetilde{X}_r \cong X$; otherwise, $X_r \cong \widetilde{X}_{\mathbb{Z}}$ is the two-sheeted cover of X in (11.2). □

Theorem 11.10. *Let X be a closed connected n-manifold. Then*

- *If X is $\mathcal{R}$-orientable, then the map $H_n(X, \mathcal{R}) \longrightarrow H_n(X|x, \mathcal{R})$ is an isomorphism for all $x \in X$.*
- *If X is not $\mathcal{R}$-orientable, then the map $H_n(X, \mathcal{R}) \longrightarrow H_n(X|x, \mathcal{R}) \cong \mathcal{R}$ is injective with image the set of 2-torsion points $\{r \in \mathcal{R} \colon 2r = 0\}$.*
- *$H_i(X, \mathcal{R}) = 0$ for $i > n$.*

Proof. The first two statements follow from the description of $X_{\mathcal{R}}$ above and the lemma below. The third one is a specialization of the second statement in the lemma below. □

Definition 11.11. An element of $H_n(X, \mathcal{R})$ whose image in $H_n(X|x, \mathcal{R})$ is a generator for all $x \in X$ is called a **fundamental class** of X and is denoted by $[X]$.

Exercise 11.12. Show that if X admits a fundamental class, then X is compact.

The following is a similar statement for non-compact manifolds.

Lemma 11.13. *Let X be an n-manifold and $Y \subset X$ be a compact subset. Then*

- *If X is $\mathcal{R}$-orientable, then a section $x \longrightarrow \mu_x$ of $X_{\mathcal{R}}$ orienting X determines a unique* ***relative fundamental class*** *$[X|Y] \in H_n(X|Y, \mathcal{R})$ whose image in $H_n(X|x, \mathcal{R})$ is μ_x for all $x \in Y$.*
- *$H_i(X|Y, \mathcal{R}) = 0$ for $i > n$.*

Proof. Proof uses Mayer-Vietoris long exact sequence. □

Lemma 11.14. *If X is a closed connected n-manifold, the $H_{n-1}(X,\mathbb{Z})_{\rm tor}$ is trivial iff X is orientable and is $\mathbb{Z}_2$ if X is non-orientable.*

Proof. The result follows from the Universal Coefficient Theorem for homology; See Remark 9.22. We have a short exact sequence

$$0 \longrightarrow \big(H_n(X,\mathbb{Z})\otimes\mathcal{R}\big) \longrightarrow H_n(X,\mathcal{R}) \longrightarrow \mathrm{Tor}^1(H_{n-1}(X,\mathbb{Z}),\mathcal{R}) \longrightarrow 0\ .$$

If X is orientable, then it is $\mathcal{R}$-orientable and the first map is an isomorphism. Since $\mathrm{Tor}^1(H_{n-1}(X,\mathbb{Z}),\mathcal{R})$ depends (non-trivially) only on $H_{n-1}(X,\mathbb{Z})_{\rm tor}$. We conclude that $H_{n-1}(X,\mathbb{Z})_{\rm tor}=0$. If X is not orientable, then the first term is zero and $H_n(X,\mathbb{Z}_m)$ is either $\mathbb{Z}_2$ or 0 depending on whether m is even or odd. This forces $H_{n-1}(X,\mathbb{Z})_{\rm tor}=\mathbb{Z}_2$. □

Exercise 11.15. Prove the last statement of Theorem 11.10 for non-compact X.

Since every n-manifold is constructed by gluing pieces that are homeomorphic to open sets in $\mathbb{R}^n$, it is natural to consider Mayer-Vietoris long-exact sequence for inductively proving the Poincare duality statement of Theorem 11.1. However, this idea would involve dealing with open (non-closed) manifold. We will define and use compactly supported cohomology groups for this purpose.

The definition of singular chain complex involves finite sums of singular simplices. Thus, every singular chain in $C_k(X)$ is compactly supported in X. On the other hand, the cochain groups $C^k(X,\mathcal{R})$ are defined to be the dual spaces $\mathrm{Hom}(C_k(X),\mathcal{R})$. The elements of $\mathrm{Hom}(C_k(X),\mathcal{R})$ do not need to be compactly supported in the following sense. In fact, $1\in C^0(X,\mathbb{Z})$ which corresponds to constant function $1\colon X\longrightarrow\mathbb{Z}$ is compactly supported iff X is compact.

Definition 11.16. We say $\varphi\in C^k(X,\mathcal{R})$ is compactly supported iff there is a compact set $K\subset X$ such that φ is non-zero on $\sigma\in C_k(X)$ only if $\sigma\in C_k(K)$. We denoted the subgroup of compactly supported cochains by $C^\bullet_c(X,\mathcal{R})$.

Obviously, if X is compact, then every cochain is compactly-supported. Also, it is clear that $(C^\bullet_c(X,\mathcal{R}),\partial^\bullet)$ is a sub-cochain complex of $(C^\bullet(X,\mathcal{R}),\partial^\bullet)$. The cohomology groups of $(C^\bullet_c(X,\mathcal{R}),\partial^\bullet)$ are called the compactly-supported singular cohomology groups of X and are denoted by $H^\bullet_c(X,\mathcal{R})$. Compactly supported simplicial and CW cohomology groups can be defined similarly and are isomorphic to the compactly supported singular cohomology groups.

Example 11.17. Calculate $H^0_c(\mathbb{R},\mathbb{Z})$ and $H^1_c(\mathbb{R},\mathbb{Z})$.

Proposition 11.18. *For every topological space X we have*

$$H^\bullet_c(X)=\varinjlim\ H^\bullet(X|K)$$

where the direct limit is over compact sets $K\subset X$ and with respect to the pullback maps

$$H^\bullet(X|K)\longrightarrow H^\bullet(X|K')$$

whenever $K\subset K'$ are compact. Similarly, if $X=\bigcup_{i=1}^\infty X_i$ such that $X_i\subset X_{i+1}$ and each compact set is contained in some X_i, then

$$H_\bullet(X)=\varinjlim\ H_\bullet(X_i)$$

Example 11.17 and its extension below show that, unlike homology and cohomology, the compactly supported cohomology is not an invariant of homotopy type.

Exercise 11.19. Calculate compactly supported cohomology groups of $\mathbb{R}^n$.

Exercise 11.20. If $X = U \cup V$ is a union of open sets, prove that the Mayer-Vietoris long-exact sequence of compactly supported cohomology has the following form

$$\cdots \longrightarrow H_c^\ell(U\cap V) \longrightarrow H_c^\ell(U) \oplus H_c^\ell(V) \longrightarrow H_c^\ell(X) \longrightarrow H_c^{\ell+1}(U\cap V) \longrightarrow \cdots .$$

Thus, compared to the MV LES for the ordinary cohomology groups, in each degree ℓ, the three terms appear in the opposite order.

The definition of cap product extends to a pairing between relative homology and cohomology groups by

$$\frown: H_k(X,Y,\mathcal{R}) \otimes_{\mathcal{R}} H^\ell(X,\mathcal{R}) \longrightarrow H_{k-\ell}(X,Y,\mathcal{R});$$
$$\frown: H_k(X,Y,\mathcal{R}) \otimes_{\mathcal{R}} H^\ell(X,Y,\mathcal{R}) \longrightarrow H_{k-\ell}(X,\mathcal{R}).$$

Now, suppose X is a possibly non-compact $\mathcal{R}$-orientable n-manifold. For compact sets $K' \subset K$ we have a diagram

$$\begin{tikzcd} H_n(X|K,\mathcal{R}) \otimes_{\mathcal{R}} H^\ell(X|K,\mathcal{R}) \arrow[r, "\frown"] \arrow[d, "\iota_* \quad \iota^*"'] & H_{n-\ell}(X,\mathcal{R}) \arrow[d, "\cong"] \\ H_n(X|K',\mathcal{R}) \otimes_{\mathcal{R}} H^\ell(X|K',\mathcal{R}) \arrow[r, "\frown"] & H_{n-\ell}(X,\mathcal{R}) \end{tikzcd},$$

where ι is the inclusion map of pairs $(X, X-K) \longrightarrow (X, X-K')$. By Lemma 11.13, there are unique relative fundamental classes $[X|K] \in H_n(X|K,\mathcal{R})$ and $[X|K'] \in H_n(X|K',\mathcal{R})$ restricting to the given local orientations at each point of K and K', respectively. The uniqueness implies that $\iota_*[X|K] = [X|K']$. Therefore, the naturality of cap product implies that

$$[X|K'] \frown \alpha = [X|K] \frown \iota^*\alpha \qquad \forall\ \alpha \in H^\ell(X|K',\mathcal{R}).$$

For each K, let

$$\mathrm{PD}_K \colon H^\ell(X|K,\mathcal{R}) \longrightarrow H_{n-\ell}(X,\mathcal{R}), \qquad \alpha \longrightarrow [X|K] \frown \alpha \qquad \alpha \in H^\ell(X|K,\mathcal{R}).$$

By the argument above, for $K' \subset K$ the following diagram commutes:

$$\begin{tikzcd} H^\ell(X|K,\mathcal{R}) \arrow[r, "\mathrm{PD}_K"] & H_{n-\ell}(X,\mathcal{R}) \\ H^\ell(X|K',\mathcal{R}) \arrow[u, "\iota^*"] \arrow[ur, "\mathrm{PD}_{K'}"] & \end{tikzcd}$$

Letting K get larger and larger, by Proposition 11.18, in the limit we get a well-defined Poincare duality map

$$\mathrm{PD}_X \colon H_c^\ell(X,\mathcal{R}) = \varinjlim H^\ell(X|K,\mathcal{R}) \longrightarrow H_{n-\ell}(X,\mathcal{R}).$$

We will prove the following generalization of the second statement in Theorem 11.1.

Theorem 11.21. *Suppose $\mathcal{R}$ is a commutative ring. If X is a possibly non-compact $\mathcal{R}$-orientable n-manifold, then*

$$\mathrm{PD}_X\colon H^\ell_c(X,\mathcal{R}) \longrightarrow H_{n-\ell}(X,\mathcal{R}) \tag{11.3}$$

is an isomorphism for all ℓ.

Proof. The proof uses Mayer-Vietoris long exact sequence and the naturality properties of $[X|K]$ and $\frown$. Here is a sketch of the proof.

Lemma 11.22. *If the $\mathcal{R}$-orientable manifold X is a union of two open sets U and V, then the PD maps in (11.3) for different degrees complete the homology and cohomology Mayer-Vietoris long exact sequences into the following commutative diagram*

$$\begin{tikzcd}
\cdots \arrow[r] & H^\ell_c(U\cap V) \arrow[r] \arrow[d, "\mathrm{PD}_{U\cap V}"] & H^\ell_c(U)\oplus H^\ell_c(V) \arrow[r] \arrow[d, "\mathrm{PD}_U \quad \mathrm{PD}_V"] & H^\ell_c(X) \arrow[r] \arrow[d, "\mathrm{PD}_X"] & H^{\ell+1}_c(U\cap V) \arrow[r] & \cdots \\
\cdots \arrow[r] & H_{n-\ell}(U\cap V) \arrow[r] & H_{n-\ell}(U)\oplus H_{n-\ell}(V) \arrow[r] & H_{n-\ell}(X) \arrow[r] & H_{n-\ell-1}(U\cap V) \arrow[r] & \cdots
\end{tikzcd}$$

Furthermore, if PD_U, PD_V, and $\mathrm{PD}_{U\cap V}$ are isomorphisms, then PD_X is also an isomorphism.

Proof. Sketch of the proof.

1. For compact sets $K\subset U$ and $L\subset V$, replace $H^\ell_c(U)$ with $H^\ell(U|K)$, $H^\ell_c(V)$ with $H^\ell(V|L)$, $H^\ell_c(U\cap V)$ with $H^\ell(U\cap V \mid K\cap L)$, and $H^\ell_c(X)$ with $H^\ell(X\mid K\cup L)$.
2. Furthermore, by Excision, replace $H^\ell(U\cap V\mid K\cap L)$ with $H^\ell(X\mid K\cap L)$ and $H^\ell(U|K)\oplus H^\ell(V|L)$ with $H^\ell(X|K)\oplus H^\ell(X|L)$.
3. Prove, directly from the definition of PD, that the resulting diagram is commutative.
4. Pass to the limit.

The last statement of the lemma follows from Five-Lemma. □

We finish the proof of Theorem 11.21 by exhausting X with a sequence of open sets $U_1\subset U_2\subset \cdots$ such $\bigcup_{i=1}^\infty U_i = X$ and induction on i. Such a sequence can be constructed by putting a metric on X and defining $U_i=\{x\in X\colon \mathrm{dist}(x,x_0)<i\}$ for some fixed point $x_0\in X$. □

Exercise 11.23. Show that every manifold is a metric space (i.e., the topology is a metric topology).

Example 11.24. If X is a closed oriented n-manifold, then

$$H^i(X,\mathbb{Z})\cong H_{n-i}(X,\mathbb{Z}) \qquad \text{and}$$
$$H^i(X,\mathbb{Z})\cong H_i(X,\mathbb{Z})_{\mathrm{free}}\oplus H_{i-1}(X,\mathbb{Z})_{\mathrm{tor}}.$$

Therefore,

$$H_{n-i}(X,\mathbb{Z})_{\mathrm{free}}\cong H_i(X,\mathbb{Z})_{\mathrm{free}} \quad \text{and} \quad H_{n-i}(X,\mathbb{Z})_{\mathrm{tor}}\cong H_{i-1}(X,\mathbb{Z})_{\mathrm{tor}}.$$

In particular, if n is odd then $\chi(X)=0$.

12 Homotopy

TBW

Math 6400 Take-Home Final, due December 18 by email

No collaboration! Feel free to browse Hatcher and internet.

- Q1. For $n < m$ and $i \leq m$, what are the relative homotopy groups $\pi_i(\mathbb{RP}^m, \mathbb{RP}^n)$.
- Q2. For $n < m$, compute the relative homology groups $H_i(\mathbb{RP}^m, \mathbb{RP}^n)$ with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients.
- Q3. Calculate the integral cohomology groups with compact support of an n-punctured Riemann surface of genus g.
- Q4. Show that a CW complex is contractible (to a point) if it is the union of an increasing sequence of sub-complexes
$$X_1 \subset X_2 \subset \cdots$$
such that each inclusion $X_i \subset X_{i+1}$ is null-homotopic (i.e. X_i is contractible in X_{i+1}). Use this to prove that
$$S^\infty = \Big\{x = (x_1, \ldots, x_n, 0, 0, \cdots) \colon n > 0, \sum_{i=1}^{n} x_i^2 = 1\Big\}$$
is contractible.
- Q5. Let M be a closed, connected, orientable n-dimensional manifold, and suppose that there is a continuous map $f \colon S^n \longrightarrow M$ such that the induced homomorphism
$$f_* \colon H_n(S^n, \mathbb{Z}) \longrightarrow H_n(M, \mathbb{Z})$$
is non-trivial. Show that $H_k(M, \mathbb{Z})$ is torsion for all $0 < k < n$.
- Q6. Show that a compact manifold does not retract onto its boundary.

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